RATIONAL APPROXIMATIONS OF TRANSFER FUNCTIONS OF SOME VISCOELASTIC RODS, WITH APPLICATIONS TO ROBUST CONTROL

HELSINKI UNIVERSITY OF TECHNOLOGY
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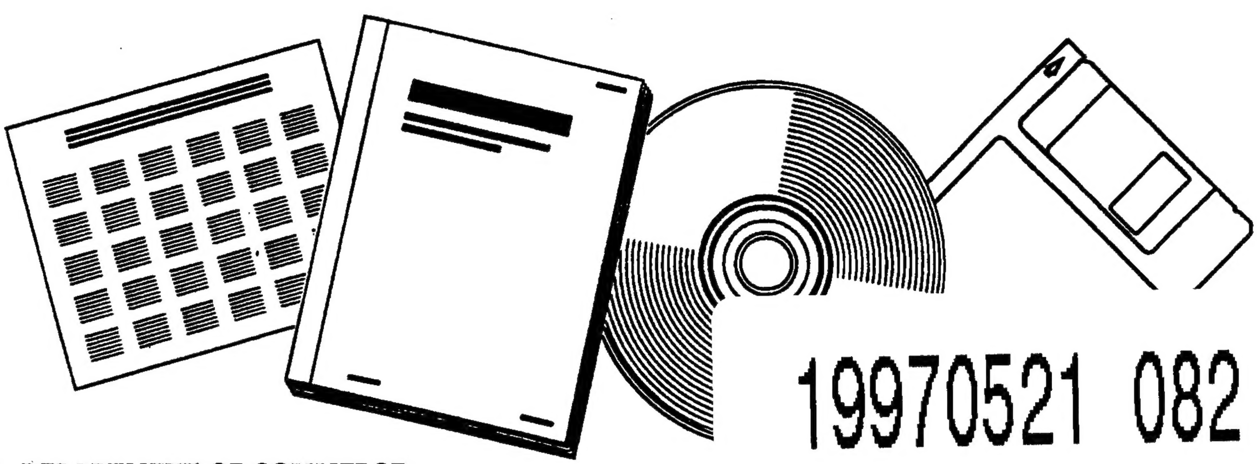

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Authors: K. B. Hannsgen, O. J. Staffans, and R. L. Wheeler.

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RATIONAL APPROXIMATIONS OF TRANSFER FUNCTIONS OF SOME VISCOELASTIC RODS, WITH APPLICATIONS TO ROBUST CONTROL

KENNETH B. HANNSGEN,* OLOF J. STAFFANS,† AND ROBERT L. WHEELER*

*DEPARTMENT OF MATHEMATICS
VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
BLACKSBURG, VA 24061-0123, USA
AND †INSTITUTE OF MATHEMATICS
HELSINKI UNIVERSITY OF TECHNOLOGY

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Kenneth B. Hannsgen,* Olof J. Staffans† and Robert L. Wheeler*: Rational Approximations of Transfer Functions of Some Viscoelastic Rods, with Applications to Robust Control. *Department of Mathematics, Virginia Polytechnic Institute and State University, †Helsinki University of Technology, Institute of Mathematics, Research Reports A310 (1992)

Abstract. We study rational approximations of the transfer function $\widehat{P}$ of a uniform or nonuniform viscoelastic rod undergoing torsional vibrations that are excited and measured at the same end. The approximation is to be carried out in a way that is appropriate, with respect to stability and performance, for the construction of suboptimal rational stabilizing compensators for the rod. The function $\widehat{P}$ can be expressed as $\widehat{P}(s) = s^{-2}g(\beta^2(s))$, where g is an infinite product of fractional linear transformations and β is a (generally transcendental) function that characterizes a particular viscoelastic material. First, $g(\beta)$ is approximated by its partial products $g_N(\beta)$. For relevant values of β, convergence rates for g_N are analyzed in detail. Convergence suitable for our problem requires the introduction of a new irrational convergence factor, which must be approximated separately. In addition, the fractional linear factors in $\beta^2(s)$ that appear in $g_N(\beta(s))$ must be replaced by something rational. When the damping is weak it is possible to do this by separating the oscillatory modes from the "creep" modes and ignoring the latter; in general, this step remains incomplete. Some numerical data illustrating all the stages of the process as well as the final results for various viscoelastic constitutive relations are presented.

Keywords: rational approximation, viscoelastic rod, robust control

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Helsinki University of Technology, Institute of Mathematics
Otakaari 1, SF-02150 Espoo, Finland

1. Introduction. We examine a scalar input-output system that models a boundary feedback scheme for the damping of torsional vibrations in a (not necessarily uniform) cylindrical rod of circular cross section, consisting of a linear viscoelastic material. The open loop transfer function for the system is irrational, and we study the problem of approximating some ideal compensator by a proper rational one. The approach is in the spirit of [12], where bending vibrations in an Euler-Bernoulli beam with Kelvin-Voigt damping were studied (and where a discussion of potential applications of the method is found). In particular, a compensator derived from the full distributed parameter model is approximated. The general method is designed to deal with a wide range of viscoelastic materials and structures. Here we treat the full range of linear viscoelastic constitutive relations, but we examine only one structure, namely, the case of torsional vibrations (i.e. the viscoelastic wave equation) with actuator and sensor collocated at one end of the rod. An analogous study for bending motion in beams will follow in a separate paper [10]; for non-collocated sensor and actuator, new issues arise that will not be addressed here, see [12]. We remark that in addition to the work in [12], the problem of robust controller design for an Euler-Bernoulli beam with strong damping, namely Kelvin-Voigt damping, has been considered in several papers; in particular, we cite [11], [3] and [4].

In the particular (collocated) cases that we discuss, the open-loop transfer functions $\widehat{P}(s)$ have no zeros or poles in the open right half-plane, and no zeros or poles on the imaginary axis apart from a pole at zero and a (fractional order) zero at infinity. The transfer function of a typical "optimal" compensator will be some rational function divided by $\widehat{P}(s)$; since the latter function is irrational, the compensator obtained in this way will be irrational too. In order to get a compensator that can be physically implemented, one has to approximate $\widehat{P}(s)$ by a rational function, preferably of low degree.

As a model problem illustrating the type of approximation we will make, consider the equation

$$w_{tt}(x,t) + 2aw_t(x,t) + a^2 w(x,t) = w_{xx}(x,t) \qquad (0 < x < 1, 0 < t < \infty, a > 0) \tag{1}$$

with boundary and initial conditions

$$w_x(0,t) = 0, w_x(1,t) = u(t) = \text{input}, \; w(x,0) = w_t(x,0) = 0$$

and output $w(1,t)$. This is a wave equation with dispersion and viscous damping, and it is not formally covered by our results, but it is quite similar in nature. Solution of the system for the Laplace transform

$$\widehat{w}(x,s) = \int_0^\infty \mathrm{e}^{-st} w(x,t)\,\mathrm{d}t$$

leads to the formula $\widehat{w}(1,s) = \widehat{P}(s)\widehat{u}(s)$, where the transfer function is

$$\widehat{P}(s) = \frac{\cosh(s+a)}{(s+a)\sinh(s+a)} = \frac{1}{(s+a)^2}\prod_{k=1}^{\infty}\frac{1+(s+a)^2/\xi_k^2}{1+(s+a)^2/\eta_k^2} \tag{2}$$

with $\xi_k = (2k-1)\pi/2$, $\eta_k = k\pi$. We attempt to approximate $\widehat{P}$ with a finite product

$$\widehat{P}_N(s) \equiv \frac{1}{(s+a)^2} \prod_{k=1}^{N} \frac{1+(s+a)^2/\xi_k^2}{1+(s+a)^2/\eta_k^2}. \tag{3}$$

As we shall see in Section 2, this approximation should be done in such a way that, e.g., $(\widehat{P}(s)/\widehat{P}_N(s)-1)\widehat{T}(s)$ tends to zero uniformly in the right half-plane; the function $\widehat{T}$ is a certain weight function (a design parameter) that has a zero at infinity.

In a uniform or nonuniform viscoelastic rod model one gets a similar transfer function. The main difference is that the factor $(s+a)$ is replaced by a (possibly) transcendental function $\beta(s)$ which reflects the viscoelastic memory kernel for the material.

The geometry of the problem remains the same for all different functions $\beta(s)$, so the same partial product scheme is used for all materials. We prove convergence of these partial products in regions of the β-plane that include the range of $\beta(s)$. The detailed mechanical properties of the viscoelastic material enter solely through $\beta(s)$ and will influence the number of factors needed for the desired degree of accuracy in the partial product.

The product expansion that we use is, in principle, well-known. The functions that need to be approximated have alternating zeros and poles on straight lines in the complex plane. In particular, in the case of the rod these zeros and poles lie on the imaginary axis. According to the Hermite-Biehler theorem, functions of this type can be represented as infinite products of linear fractional transformations. However, the Hermite-Biehler theorem itself does not give a bound on the behavior of the error terms at infinity, and a substantial part of this work is devoted to the development of a precise description of the asymptotic error at infinity (Section 4). In particular, we show that one gets a better approximation at infinity if instead of simply truncating the infinite product expansion, we also multiply the truncated product by an additional irrational factor (see the discussion preceding equation (31) in Section 4).

To complete the construction of a rational approximation of $\widehat{P}$ for models where β^2 is irrational we have to approximate linear fractional transformations of $\beta^2(s)$ by linear fractional transformations of s. In Section 5 we suggest a very simple, low order approximation which in many cases leads to satisfactory results. The idea behind this approximation is to separate the dynamic modes from the creep modes, and to ignore the latter type of modes. This seems to work quite well when the internal structural damping is small.

Finally, in Section 6 we discuss the extra irrational convergence factor that was introduced in Section 4, formula (31). In particular, we show with asymptotic estimates and numerical experiments that we get reasonably good results by simply ignoring the extra factor completely, provided that the design parameters $\widehat{S}$ and $\widehat{T}$ are properly chosen.

The authors thank Professor Joseph A. Ball for several helpful discussions.

2. Weighted Rational Approximations of the Optimal Compensator. As we shall see in Section 3, the transfer functions that we deal with here can be expressed as the quotient of two outer H^∞-functions. More precisely, for the type of rod problem treated here, they are continuous and have no zeros or poles in the closed right half-plane, except for a zero at infinity (not of integer order, in general) and a pole of order two at zero. (In the essentially identically situation where the left end of the rod is fixed rather than free, the transfer function is regular at zero.) In particular, this means that by the appropriate choice of compensator, one can achieve almost any sensitivity function $\widehat{S} = (1+\widehat{P}\widehat{C})^{-1}$. The only constraints that have to be imposed on $\widehat{S}$ are some growth conditions at infinity, and possibly at zero.

Within this class of feasible sensitivity functions $\widehat{S}$, one would like to find an optimal one. There are several different methods available, including mixed sensitivity H^∞-optimization and H^2-optimization with H^∞ constraints. It may happen, as in [18], that $\widehat{S}$ itself is already a suboptimal approximation of an ideal sensitivity function that fails to satisfy the stability conditions (4) below. Here we shall not go into this optimization procedure. Instead we assume that one has found, in one way or another, a sensitivity function $\widehat{S}$ that one would like the closed loop system to have.

As we mentioned above, not every $\widehat{S}$ is permissible, because the open loop transfer function $\widehat{P}$ has a zero at infinity, and possibly a pole at zero. This can be seen in many ways, one of which is the following. The closed-loop system is always required to be stable, i.e., the four closed-loop transfer functions

$$\begin{aligned}
\widehat{S} &= \frac{1}{1+\widehat{P}\widehat{C}},\\
\widehat{T} &= 1-\widehat{S} = \widehat{P}\widehat{C}\widehat{S} = \frac{\widehat{P}\widehat{C}}{1+\widehat{P}\widehat{C}},\\
\widehat{C}\widehat{S} &= \frac{\widehat{C}}{1+\widehat{P}\widehat{C}},\\
\widehat{P}\widehat{S} &= \frac{\widehat{P}}{1+\widehat{P}\widehat{C}},
\end{aligned}$$

should all belong to H^∞, and their H^∞-norms should not exceed certain numbers that can be given *a priori.* (These numbers depend on the sizes of the sensor and actuator noises, etc.) In particular, we have

$$\|\widehat{C}\widehat{S}\|_{H^\infty} \le K_C, \qquad \|\widehat{P}\widehat{S}\|_{H^\infty} \le K_P, \tag{4}$$

for some constants K_C and K_P. For our scalar system the former of these conditions is equivalent to

$$|1-\widehat{S}(s)| = |\widehat{T}(s)| \le K_C|\widehat{P}(s)|, \tag{5}$$

and the latter is equivalent to

$$|\widehat{S}(s)| \le K_P|\widehat{P}(s)|^{-1}. \tag{6}$$

Note that (5) gives

$$\widehat{S}(s) = 1 + O(|\widehat{P}(s)|) \text{ as } s \to \infty, \tag{7}$$

which combined with (4) implies

$$\limsup_{s\to\infty} |\widehat{C}(s)| \leq K_C,$$

that is, $\widehat{C}$ has to be proper. The restriction (6) comes into play mainly in those cases where $\widehat{P}$ has a pole at zero, since it implies that

$$\widehat{S}(s) = O(|\widehat{P}(s)|^{-1}) \text{ as } s \to 0. \tag{8}$$

The conditions (4) imply stability. To see this, observe that

$$\widehat{S}\widehat{T} = (\widehat{C}\widehat{S})(\widehat{P}\widehat{S});$$

hence, for each s,

$$|\widehat{S}(s)|(|\widehat{S}(s)| - 1) \leq K_C K_P,$$

and

$$|\widehat{S}(s)| \leq 1/2 + \sqrt{1/4 + K_C K_P}.$$

A similar estimate holds for $|\widehat{T}(s)|$.

In the sequel we assume that the "optimal" sensitivity function $\widehat{S}$ that we would like to implement is feasible in the sense that it satisfies (4).

Once a sensitivity function $\widehat{S}$ is given, one may compute the corresponding optimal compensator from the formula

$$\widehat{C} = \frac{1 - \widehat{S}}{\widehat{S}\widehat{P}} = \frac{\widehat{T}}{\widehat{S}\widehat{P}}.$$

There is only one thing wrong with this $\widehat{C}$; in general it will not be rational, due to the fact that $\widehat{P}$ is not rational, and it cannot be implemented exactly by means of standard circuits. This leads to the main point of this paper. Suppose that we have a given $\widehat{S}$ satisfying (4). How can we find a *rational* compensator $\widehat{C}_N$ such that the sensitivity function $\widehat{S}_N$ corresponding to this compensator will be reasonably close to the ideal one $\widehat{S}$, and will satisfy the analogue of (4), namely,

$$\|\widehat{C}_N \widehat{S}_N\|_{H^\infty} \leq K_C, \qquad \|\widehat{P}\widehat{S}_N\|_{H^\infty} \leq K_P, \tag{9}$$

as well? We shall assume that the optimal $\widehat{S}$ (hence the optimal $\widehat{T}$) is rational. Thus, it is really a question of approximating $\widehat{P}$ by a rational function $\widehat{P}_N$, and setting $\widehat{C}_N = \widehat{T}/(\widehat{S}\widehat{P}_N)$.

There are several ways that one may interpret the words "reasonably close" in the preceding paragraph, and to each different way corresponds a different measure of how good the approximation is. Perhaps one of the most natural measurements of the goodness of an approximating compensator $\widehat{C}_N$ is the smallness of either the H^2-norm or the H^∞-norm of $\widehat{S}_N - \widehat{S}$. These norms can be written in the form

$$\begin{aligned}\|\widehat{S}_N - \widehat{S}\|_{H^p} &= \left\| \frac{1}{1+\widehat{C}_N\widehat{P}} - \frac{1}{1+\widehat{C}\widehat{P}} \right\|_{H^p} \\ &= \left\| (\tau_N - 1) \frac{1}{(1+\widehat{C}_N\widehat{P})} \frac{\widehat{C}\widehat{P}}{(1+\widehat{C}\widehat{P})} \right\|_{H^p},\end{aligned}$$

where $p = 2$ or $p = \infty$, and

$$\tau_N = \widehat{C}_N / \widehat{C} = \widehat{P} / \widehat{P}_N.$$

If we ignore higher order terms, then this becomes $\|(\tau_N - 1)\widehat{S}\widehat{T}\|_{H^p}$. Thus, in this case the problem could be interpreted as finding, for each N, a controller $\widehat{C}_N$ of order N, say, that minimizes

$$\|\widehat{S}_N - \widehat{S}\|_{H^p} \simeq \left\|(\tau_N - 1)\widehat{S}\widehat{T}\right\|_{H^p}.$$

In other words, we get a weighted approximation problem, where the weight is $\widehat{S}\widehat{T}$. Because of (7), this weight function will have a zero at infinity, and, if $\widehat{P}$ has a pole at zero, then, because of (8), it will also have a zero at zero.

In the discussion above we have ignored all higher order terms and the stability bounds (9). When these are taken into account one gets some additional constraints. The exact expression for $\widehat{S}_N - \widehat{S}$ is

$$\widehat{S}_N - \widehat{S} = (1 - \tau_N)\widehat{T}\widehat{S}_N,$$

where $\widehat{S}_N$ can be written in the form

$$\widehat{S}_N = \frac{\widehat{S}}{1 + (\tau_N - 1)\widehat{T}}. \tag{10}$$

Thus, we have the more precise estimate (for $p = 2$ or $p = \infty$)

$$\|\widehat{S}_N - \widehat{S}\|_{H^p} \le \left\|(\tau_N - 1)\widehat{S}\widehat{T}\right\|_{H^p} \left\|\left(1 + (\tau_N - 1)\widehat{T}\right)^{-1}\right\|_{H^\infty}.$$

To make this norm small it suffices to minimize the H^p-norm of $(\tau_N - 1)\widehat{S}\widehat{T}$ subject to the constraint that for some constant $\alpha < 1$,

$$\left\|(\tau_N - 1)\widehat{T}\right\|_{H^\infty} \le \alpha < 1. \tag{11}$$

This is the same problem as before, except for the additional constraint (11). Of course, one could also use the fact that

$$\left\|(\tau_N - 1)\widehat{S}\widehat{T}\right\|_{H^p} \le \left\|(\tau_N - 1)\widehat{T}\right\|_{H^p} \left\|\widehat{S}\right\|_{H^\infty},$$

and minimize the H^p-norm of $(\tau_N - 1)\widehat{T}$ instead of the H^p-norm of $(\tau_N - 1)\widehat{S}\widehat{T}$ since the H^∞-norm of $\widehat{S}$ is finite.

Let us still take a closer look at the stability bounds (9). Because of (4), (10) and (11), we have

$$\begin{aligned}\left\|\widehat{C}_N\widehat{S}_N\right\|_{H^\infty} &\le \left\|\tau_N\widehat{C}\widehat{S}\right\|_{H^\infty}\left\|\left(1+(\tau_N-1)\widehat{T}\right)^{-1}\right\|_{H^\infty}\\ &\le (1-\alpha)^{-1}\left\|\tau_N\widehat{C}\widehat{S}\right\|_{H^\infty} \le \frac{K_C}{1-\alpha}\left\|\tau_N\right\|_{H^\infty},\end{aligned}$$

and

$$\left\|\widehat{P}\widehat{S}_N\right\|_{H^\infty} \le \left\|\widehat{P}\widehat{S}\right\|_{H^\infty}\left\|\left(1+(\tau_N-1)\widehat{T}\right)^{-1}\right\|_{H^\infty} \le \frac{K_P}{1-\alpha}.$$

Thus, our problem becomes the one of minimizing the H^p-norm of either $(\tau_N - 1)\widehat{S}\widehat{T}$ or $(\tau_N - 1)\widehat{T}$ subject to the constraint (11), combined with either

$$\|\tau_N\|_{H^\infty} \le \beta \tag{12}$$

or

$$\left\|\tau_N\widehat{C}\widehat{S}\right\|_{H^\infty} \le K_C\beta, \tag{13}$$

where $\beta > 1$ is some constant. Note that (12) can be much more conservative than (13) due to the fact that whenever the function $\widehat{C}\widehat{S}$ has a zero at infinity it is possible to allow τ_N to be quite large or even infinite at infinity; this is true whenever $\widehat{C}$ is strictly proper.

If one chooses some other criterion for goodness of the approximation, then one gets some other weighted approximation problem, either in H^2 or in H^∞, with additional H^∞ constraints. See, e.g., [1]. These weights will always have a zero at infinity, and sometimes a zero at zero, and H^∞-constraints of the type (11) and (13) are always present, either explicitly or implicitly, since they are intimately connected to the stability of the feedback system.

The method that we propose will not be optimal with respect to any of these minimization problems; indeed, the approximate plant $\widehat{P}_N$ will not be characterized as a minimizer. The method will, however, give fairly good results with respect to all measures of closeness of the type described. Specifically, we shall construct rational approximates $\widehat{P}_N$ (of orders that will increase linearly in N) in such a way that

- $\tau_N = \widehat{C}_N/\widehat{C} = \widehat{P}/\widehat{P}_N \to 1$ uniformly on compact subsets of the right half-plane, and
- $\limsup_{N\to\infty}\|\tau_N\widehat{C}\widehat{S}\|_{H^\infty} < \infty$.

It is easy to show that, for the plants that we consider, and for any compensator $\widehat{C}$ satisfying the stability bounds (4), we will then have $\|(\tau_N - 1)\widehat{S}\widehat{T}\|_{H^p} \to 0$ and $\|(\tau_N - 1)\widehat{T}\|_{H^p} \to 0$ for $p = 2$ and $p = \infty$ (note that $\widehat{C}(s)\widehat{S}(s) \sim \widehat{C}(s)$ and $\widehat{T}(s) \sim \widehat{C}(s)\widehat{P}(s)$ as $|s| \to \infty$). In particular, (11) will be satisfied for large values of N with

arbitrarily small α, and (13) will hold. Of course, our compensators will not have the lowest possible order corresponding to a given accuracy, but the order may be further reduced by means of some standard order reduction scheme.

As we shall see, we will not quite achieve the goal set forth. First, if $\beta^2(s)$ is not rational, our method of constructing $\widehat{P}_N$ (Section 5) leaves a residual error due to some continuous creep modes (singularities other than poles on the negative real axis) that we ignore. Although this residual error does not tend to zero as $N \to \infty$, it will be quite small in the case where the material is "nearly" elastic.

Second, our best approximations involve multiplication of the basic partial product by a new factor, irrational in β^2 and unbounded as $s \to \infty$. In Section 6 we analyze the effect of dropping or simplifying this factor; the same type of convergence ($N \to \infty$) holds, but the convergence rates are weakened. A detailed investigation of low order rational approximates for this factor remains to be done.

At each stage of the approximation, we introduce a new relative error, $\tau_N^{(1)}$ in Section 4, $\tau_N^{(2)}$ in Section 5 and $\tau_N^{(3)}$ in Section 6, and each of these sections will close with some plots of $|\tau_N^{(j)}(i\omega) - 1|$ for some model kernels. The cumulative relative error resulting from making all the approximations is then $\tau_N = \tau_N^{(1)}\tau_N^{(2)}\tau_N^{(3)}$.

We now proceed to give a detailed description of our method.

3. The Physical Models and their Transfer Functions. We study torsional vibrations in a cylindrical rod of circular cross section consisting of a viscoelastic material, for which one modifies the usual equations of motion from linear elasticity theory by permitting the stress at any point to depend on the history of the strain rate at that point. The memory dependence is governed by a relaxation function $A(t) = E + a(t)$, where $E > 0$ and $a(t)$ is completely monotonic with $0 = a(\infty) < a(0+) \leq \infty$ and

$$\int_0^\infty e^{\delta t} a(t)\,dt < \infty \text{ for some } \delta > 0.$$

As an extreme case we also include $a(t) = a\delta(t)$ where a is a positive constant and $\delta(t)$ is the delta function (Kelvin-Voigt damping). Note that the case of pure elasticity, $a(t) = 0$, is excluded.

Reasoning as in [9], we arrive at the equation

$$\rho(x)w_t(x,t) = \int_0^t A(t-r)w_{xx}(x,r)\,dr \qquad (0 < x < 1,\ t > 0). \tag{14}$$

Here (after scaling) $w(x,t)$ represents the torsion at position x along the rod at time t. The density function $\rho(x)$ is assumed to be smooth (C^3) and strictly positive in $[0,1]$. We assume that the rod is at rest for $t \leq 0$ and that its left end is stress-free. At the right end we give a stress input $u(t)$ via an applied torque and measure $w(1,t)$ as an output. Thus,

$$\begin{aligned} & w(x,0) = 0 \qquad (0 < x < 1), \\ & \sigma(0,t) \equiv \frac{d}{dt}\int_0^t A(t-r)w_x(0,r)dr = 0 \quad (t > 0), \\ & \sigma(1,t) \equiv \frac{d}{dt}\int_0^t A(t-r)w_x(1,r)dr = u(t) \quad (t > 0). \end{aligned} \tag{15}$$

We shall be working in the frequency domain; in other words, we study the Laplace transforms of the quantities of interest, denoted by $\widehat{a}(s)$, $\widehat{w}(x,s)$, etc. We introduce

$$\alpha(s) \equiv (s\widehat{A}(s))^{1/2} = (E + s\widehat{a}(s))^{1/2} \quad (s \in \mathbf{C} \setminus (-\infty, -\delta_0]),$$

$$\beta(s) \equiv s/\alpha(s).$$

Here and below $z^{1/2}$ is defined so that $z^{1/2} > 0$ when $z > 0$; δ_0 is the largest number such that $\delta_0 \leq \delta$ and $s\widehat{A}(s) > 0$ on $(-\delta_0, 0)$. After applying the Laplace transform to equations (14)–(15), we get

$$\begin{aligned} & \alpha^2(s)\widehat{w}_{xx}(x,s) - s^2\rho(x)\widehat{w}(x,s) = 0 \quad (0 < x < 1) \\ & \alpha^2(s)\widehat{w}_x(0,s) = 0 \quad \text{(free end condition)} \\ & \alpha^2(s)\widehat{w}_x(1,s) = \widehat{u}(s) = \text{boundary stress input}. \end{aligned} \tag{16}$$

When $\rho(x) \equiv 1$ the differential equation and the first boundary condition in (16) have the general solution

$$\widehat{w}(x,s) = c(s)\cosh\beta x \ \text{ with } \ \beta = \beta(s),$$

so we get that $\widehat{w}(1,s) = \widehat{P}(s)\widehat{u}(s)$, where the transfer function $\widehat{P}$ is given by

$$\widehat{P}(s) = \frac{\beta(s)}{s^2} f(\beta(s)) \equiv \frac{\beta(s)}{s^2} \coth \beta(s). \tag{17}$$

Notice that $f(\beta)$ has the product expansion

$$f(\beta) = \coth \beta = \frac{1}{\beta} \prod_{k=1}^{\infty} \frac{1+(\beta/\xi_k)^2}{1+(\beta/\eta_k)^2},$$

with zeros and poles, respectively, at the points

$$\beta = \pm\xi_k \mathrm{i} = \pm\frac{(2k-1)\pi}{2}\mathrm{i}, \text{ and } \beta = \eta_0 = 0,\ \beta = \pm\eta_k \mathrm{i} = \pm k\pi\mathrm{i}\ (k = 1, 2, \ldots).$$

For the case of general densities below, we have

PROPOSITION 3.1. *The transfer function for the rod problem has the form*

$$\widehat{P}(s) = \frac{\beta(s)}{s^2} f(\beta(s)), \tag{18}$$

where $f(\beta)$ *has the factorization*

$$f(\beta) = \frac{1}{\rho_0 \beta} \prod_{k=1}^{\infty} \frac{1+(\beta/\xi_k)^2}{1+(\beta/\eta_k)^2}, \quad \rho_0 = \int_0^1 \rho(x)\, \mathrm{d}x.$$

Here $0 = \eta_0 < \xi_1 < \eta_1 < \xi_2 < \eta_2 \ldots$ *with*

$$\xi_k = \frac{(2k-1)\pi}{2\rho_1} + O(1/k) \text{ and } \eta_k = \frac{k\pi}{\rho_1} + O(1/k) \qquad (k \to \infty), \tag{19}$$

and $\rho_1 = \int_0^1 \rho(x)^{1/2}\, \mathrm{d}x$.

Proof. Fix s (and hence $\beta = \beta(s)$) and let $y(x,z)$ be the solution of

$$y'' + z^2 \rho(x) y = 0,\ y(0) = 1,\ y'(0) = 0. \tag{20}$$

As in the uniform case, we can then compute

$$\begin{aligned} \widehat{w}(1,s) &= \frac{\beta^2}{s^2} \frac{y(1,i\beta)}{y'(1,i\beta)} \widehat{u}(s) \\ &\equiv \widehat{P}(s)\widehat{u}(s). \end{aligned}$$

To develop the product expansion claimed for $f(\beta) = \beta y(1, \mathrm{i}\beta)/y'(1, \mathrm{i}\beta)$, we apply [13, Thm. 1, p. 308, and Thm. 4, p. 315] to the function

$$g(z) = z y(1,z)/y'(1,z).$$

We establish that

(i) $y(1,z)$ and $y'(1,z)/z^2$ are entire and

$$\lim_{z\to 0} y'(1,z)/z^2 = -\int_0^1 \rho(x)\,\mathrm{d}x, \tag{21}$$

(ii) $y(1,z)$ and $y'(1,z)/z$ have no common zeros,
(iii) if m and n are real and z is nonreal, then $my(1,z)+ny'(1,z)/z \neq 0$, and
(iv) $g(z)$ is increasing on the real axis (on the intervals between its poles).

Once (i)–(iv) are proved, the results from [13] yield the expansion

$$g(z) = c\frac{z-a_0}{z}\prod_{k\in\mathbf{Z}\setminus\{0\}}\frac{(1-z/a_k)}{(1-z/b_k)},$$

where $c>0$ and (with $b_0=0$), $b_k<a_k<b_{k+1}$ for every integer k. Since $y(1,0)=1$, relation (21) then shows that $a_0c = 1/\int_0^1 \rho(x)\,\mathrm{d}x$.

From (20) it is clear that, aside from b_0, the a_k's and b_k's occur in positive-negative pairs $\pm\xi_k$, $\pm\eta_k$, so the desired product expansion for $f(\beta)$ follows. Finally, the asymptotic distribution of the ξ_k and η_k follows from Sturm-Liouville theory [2, p. 328], [5, p. 414].

Assertions (i) and (ii) follow from the elementary theory of differential equations. In particular, for the analyticity of $y'(1,z)/z^2$ at 0 and (21), differentiate (20) with respect to z to get

$$y_z''(x,z)+z^2\rho(x)y_z(x,z) = -2zy(x,z), \quad y_z(0,z)=y_z'(0,z)=0.$$

(We denote $\partial y/\partial x$ by y', $\partial y/\partial z$ by y_z, etc.; all the derivatives that will appear are continuous in (x,z).) Letting $z\to 0$, we get that $y_z(x,0)=0$, $0\le x\le 1$, so $\lim_{z\to 0} y'(1,z)/z = y_z'(1,0)=0$. Differentiating again in z, we get

$$y_{zz}''(x,z)+z^2\rho(x)y_{zz}(x,z) = -4z\rho(x)y_z(x,z)-2\rho(x)y(x,z),$$

$$y_{zz}(0,z)=y_{zz}'(0,z)=0.$$

Since $y(x,0)\equiv 1$, we can let $z\to 0$ to obtain $y_{zz}'(x,0)=-2\rho(x)$, so

$$\lim_{z\to 0} y'(1,z)/z^2 = \frac{1}{2}y_{zz}'(1,0) = -\int_0^1 \rho(x)\,\mathrm{d}x,$$

as asserted.

Assertion (iv) follows from Sturm-Liouville theory. From [2, p. 312, (22)], with $\theta(z)=\arctan\left(y(1,z)/y'(1,z)\right)$,

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \rho(1)\sin^2\theta+\cos^2\theta>0,\ z\in\mathrm{R},$$

so $g(z)$ is a product of two increasing functions between the zeros of $y'(1,z)$ on R.

Finally, for assertion (iii), suppose there is a $z \in \mathbf{C} \setminus \mathbf{R}$ such that $zy(1,z)/y'(1,z) = r \in \mathbf{R} \setminus \{0\}$. (By standard boundary value theory $y(1,z) = 0$ and $y'(1,z) = 0$ are impossible.) Multiply (20) by $\overline{y}$ and integrate by parts to get

$$0 = z^2 \int_0^1 \rho(x)|y(x)|^2 \,\mathrm{d}x - \int_0^1 |y'(x)|^2 \,\mathrm{d}x - \frac{z}{r}|y(1)|^2. \tag{22}$$

Now set $z = \mu + i\nu$ and separate (22) into real and imaginary parts:

$$0 = (\mu^2 - \nu^2) \int_0^1 \rho(x)|y(x)|^2 \,\mathrm{d}x - \int_0^1 |y'(x)|^2 \,\mathrm{d}x - \frac{\mu}{r}|y(1)|^2$$
$$0 = 2\mu\nu \int_0^1 \rho(x)|y(x)|^2 \,\mathrm{d}x - \frac{\nu}{r}|y(1)|^2.$$

Since $\nu \neq 0$, we can cancel ν in the second equation and substitute for $|y(1)|^2/r$ in the first to obtain

$$0 = -(\mu^2 + \nu^2) \int_0^1 \rho(x)|y(x)|^2 \,\mathrm{d}x - \int_0^1 |y'(x)|^2 \,\mathrm{d}x,$$

which is impossible. □

We remark that the asymptotic error $O(k^{-1})$ in formula (19) cannot be improved in general [15, Section 4.3].

Next, recall from the Introduction that we propose to approximate $\widehat{P}(s) = s^{-2}\beta(s)f(\beta(s))$ by rational functions in two steps, where in the first step we approximate $\beta f(\beta)$ by a rational function of β. The domain of s that is of concern is the right half-plane $\Re s \geq 0$. Thus, we need a rational approximation of f that is valid in the image of the right half-plane under β, i.e., the domain of our approximation is

$$\Pi = \{\, \beta(s) \mid \Re s \geq 0 \,\}.$$

We complete this Section by describing this set.

The set Π depends heavily on the size of the function A and its derivative in the neighborhood of zero.

LEMMA 3.2. *Under the hypotheses of this section,*

1. *(a)* $\beta(0) = 0$,
   (b) $\Re\beta(s) > 0$ *for* $\Re s \geq 0$, $s \neq 0$,
   (c) $\beta(\overline{s}) = \overline{\beta(s)}$,
   (d) $\Im\beta(\mathrm{i}\omega) \to \infty$ *as* $\omega \to \infty$,
   (e) $\Im\beta(\mathrm{i}\omega) > \Re\beta(\mathrm{i}\omega)$ *for* $\omega > 0$,
   (f) $\{\, \arg(s) \leq \pi/4 \,\} \subset \Pi \equiv \{\, \beta(s) \mid \Re s \geq 0,\ s \neq 0 \,\} \subset \{\, \arg(s) < \pi/2 \,\}$,
   (g) $\liminf_{s\to\infty} |s|^{-1/2}|\beta(s)| > 0$ *uniformly in* $\Re s \geq 0$,
   (h) $\beta(s) \sim sA(0+)^{-1/2}$ *if* $A(0+) < \infty$ *and* $\beta(s) = o(s)$ *if* $A(0+) = \infty$ *as* $s \to \infty$ *uniformly in* $\Re s \geq 0$.
2. *If* $A(0+) < \infty$, *then* $\Re\beta(\mathrm{i}\omega) = o(\Im\beta(\mathrm{i}\omega))$ *as* $\omega \to \infty$. *Thus,* Π *is not contained in any sector of the form* $\{\, \arg(s) \leq \pi/2 - \epsilon \,\}$ *for any* $\epsilon > 0$. *In addition,*

$$\Im\beta(\mathrm{i}\omega) \sim \frac{\omega}{A(0+)^{\frac{1}{2}}}, \qquad \omega \to \infty, \tag{23}$$

and $\Re\beta(\mathrm{i}\omega)$ satisfies the regularity condition

$$\begin{aligned}
&\lim_{\omega\to\infty}\inf_{c\le r\le 1}\frac{\Re\beta(\mathrm{i}r\omega)}{\Re\beta(\mathrm{i}\omega)}\ge c^2 \textit{ for each } c\in(0,1),\\
(24)\qquad &\lim_{\omega\to\infty}\inf_{1\le r}\frac{\Re\beta(\mathrm{i}r\omega)}{\Re\beta(\mathrm{i}\omega)}\ge 1.
\end{aligned}$$

3. *If $A'(0+)>-\infty$ (hence $A(0+)<\infty$), then $\Im\beta(\mathrm{i}\omega)-\omega A(0+)^{-1/2}\to 0$ and $\Re\beta(\mathrm{i}\omega)\to-\frac{1}{2}A'(0+)A(0+)^{-3/2}$ as $\omega\to\infty$. If $A'(0+)=-\infty$, then $\Im\beta(\mathrm{i}\omega)\to\infty$ and $\Re\beta(\mathrm{i}\omega)\to\infty$ as $\omega\to\infty$. In particular, in all cases, $\liminf_{\omega\to\infty}\Re\beta(\mathrm{i}\omega)>0$.*

Proof. 1. We have

$$\beta^2(s)=\frac{s^2}{\alpha^2(s)}=\frac{s^2}{E+s\widehat{a}(s)},\tag{25}$$

where [9, (2.14)] $\Re\widehat{a}(s)>0$ ($\Re s\ge 0$, $s\ne 0$), $\Im s\Im\widehat{a}(s)<0$ ($\Re s\ge 0$, $\Im s\ne 0$), and $a\in L^1(\mathbf{R}^+)$. Then (1a) is obvious, and a short calculation shows that $\Im s\Im\beta^2(s)>0$ ($\Im s\ne 0$) so that $|\arg\beta^2(\mathrm{i}\omega)|<\pi$, from which we see that $\beta(s)=(s^2/\alpha^2)^{1/2}$, so (1b) and (1c) are clear. For $s=i\omega$, $\omega>0$, (25) can be resolved into real and imaginary parts, showing that $\beta^2(i\omega)$ lies in the second quadrant, so (1e) follows. The fact that $\beta(s)\to\infty$ as $s\to\infty$ ($\Re s\ge 0$) together with (1e) imply (1d). Assertion (1f) is a consequence of (1b), together with (1a), $\beta(s)\to\infty$ as $s\to\infty$, (1c), (1e) and the argument principle. Finally, (1h) follows from (25) and [9, (2.18)], and (1g) is a consequence of (25) and the fact that $|\widehat{a}(s)|\le\widehat{a}(0)$ for $\Re s\ge 0$.

2. We have

$$\widehat{A}(\mathrm{i}\omega)=\frac{A(0+)}{\mathrm{i}\omega}+\frac{1}{\mathrm{i}\omega}\widehat{A'}(\mathrm{i}\omega),\tag{26}$$

so

$$\omega\widehat{A}(\mathrm{i}\omega)=-\mathrm{i}A(0+)+o(1)\quad(\omega\to\infty).$$

Using this and (25), we get that $\pi>\arg\beta^2(\mathrm{i}\omega)\to\pi$ ($\omega\to\infty$), and the first half of Part 2 follows.

To prove the regularity condition (24), we recall that the completely monotonic function $-A'(t)$ has a Bernstein representation [17, p. 160]

$$-A'(t)=\int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}\nu(x)\qquad(t>0),$$

where ν is a positive measure, and

$$-\widehat{A'}(\mathrm{i}\omega)=\int_0^\infty\frac{x\,\mathrm{d}\nu(x)}{x^2+\omega^2}-\mathrm{i}\omega\int_0^\infty\frac{\mathrm{d}\nu(x)}{x^2+\omega^2}\qquad(\omega\in\mathbf{R}).$$

From (25) and (26) we get

$$\beta(\mathrm{i}\omega)=\frac{\mathrm{i}\omega}{A(0+)^{\frac{1}{2}}}\left[1+\frac{\widehat{A'}(\mathrm{i}\omega)}{A(0+)}\right]^{-\frac{1}{2}},$$

and then the Taylor's expansion for $(1+z)^{-1/2}$ yields

$$\Im\beta(\mathrm{i}\omega) \sim \frac{\omega}{A(0+)^{1/2}} \text{ and } \Re\beta(\mathrm{i}\omega) \sim \frac{\omega^2}{2A(0+)^{3/2}} \int_0^\infty \frac{\mathrm{d}\nu(x)}{x^2+\omega^2}, \quad \omega \to \infty.$$

Thus, (23) holds, and (24) easily follows since $\omega^2(x^2+\omega^2)^{-1}$ is increasing in ω and $(c\omega)^2\,(x^2+(c\omega)^2)^{-1} \geq c^2\omega^2(x^2+\omega^2)^{-1}$ when $c \in (0,1)$.

3. This was proved in [9, Lemmas 2.2 and 2.3]. □

For particular examples one can very precisely describe the region Π. We shall examine closely the following model kernels (in decreasing order of structural damping):

1. $\widehat{A}_1(s) = E/s + \epsilon$; Kelvin-Voigt damping, where formally A_1 is the sum of a constant and a constant times the unit point mass at zero.
2. $A_2(t) = E + (\epsilon\delta^\mu/\Gamma(\mu))\,t^{\mu-1}\mathrm{e}^{-\delta t}$, $\widehat{A}_2(s) = E/s + \epsilon(1+s/\delta)^{-\mu}$; $0<\mu<1, \epsilon, \delta > 0, \Gamma =$ gamma function; a modified "fractional derivative" model (see [16]) of order $1-\mu$ with exponential decay as $t \to \infty$.
3. $A_3(t) = E + (\epsilon\delta^{\mu+1}/\Gamma(\mu+1))\int_t^\infty \tau^{\mu-1}\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau$,
   $\widehat{A}_3(s) = E/s + (\epsilon\delta/(\mu s))\,(1-(1+s/\delta)^{-\mu})$; an intermediate model of order $1-\mu$ with $A(0+) < \infty$ and $A'(0+) = -\infty$.
4. $A_4(t) = E + \epsilon\delta\mathrm{e}^{-\delta t}$, $\widehat{A}_4(s) = E/s + \epsilon/\,(1+s/\delta)$; standard linear solid model.

Observe that the constants have been chosen in such a way that in all cases $\widehat{A}(s) - E/s - \epsilon \to 0$ as $s \to 0$, so that the different examples have the same low frequency behavior. The constant μ is related to the behavior of A near zero, and δ represents a bandwidth constant (the transfer functions do not differ much from each other for $|s| << \delta$).

For the kernels listed above, we can use the binomial series to deduce the following precise estimates valid for large values of $|s|$ in $\Re s \geq 0$. The proofs are straightforward and are left to the reader.

LEMMA 3.3. *The following estimates are valid:*

1. *If* $\widehat{A} = \widehat{A}_1$, *then* $\beta(s) = \epsilon^{-1/2}s^{1/2} + O(|s|^{-1/2})$ *as* $|s| \to \infty$, $\Re s \geq 0$.
2. *If* $\widehat{A} = \widehat{A}_2$, *then* $\beta(s) = \delta^{-\mu/2}\epsilon^{-1/2}s^{(1+\mu)/2} + O(|s|^{(3\mu-1)/2})$ *as* $|s| \to \infty$, $\Re s \geq 0$.
3. *If* $\widehat{A} = \widehat{A}_3$, *then* $\beta(s) = (E+\epsilon\delta/\mu)^{-1/2}s + \frac{1}{2}\epsilon\delta^{\mu+1}\mu^{-1}(E+\epsilon\delta/\mu)^{-3/2}s^{1-\mu} + O(|s|^{1-2\mu})$ *as* $|s| \to \infty$, $\Re s \geq 0$.
4. *If* $\widehat{A} = \widehat{A}_4$, *then* $\beta(s) = (E+\epsilon\delta)^{-1/2}s + \frac{1}{2}\epsilon\delta^2(E+\epsilon\delta)^{-3/2} + O(|s|^{-1})$ *as* $|s| \to \infty$, $\Re s \geq 0$.

The estimates in Lemma 3.3 allow one to describe the asymptotic behavior ($\omega \to \infty$) of the curves $\beta(\pm\mathrm{i}\omega)$. In the following paragraph, C denotes a positive constant that depends on the model as well as the values of the parameters μ, E, ϵ, δ. We leave it to the interested reader to determine the value of C in each case. All estimates are valid as $\omega \to \infty$.

We have $\beta(-\mathrm{i}\omega) = \overline{\beta(\mathrm{i}\omega)}$. For A_1, $\beta(\mathrm{i}\omega) \sim \epsilon^{-1/2}\mathrm{e}^{\pi\mathrm{i}/4}\omega^{1/2}$, and for A_2, $\beta(\mathrm{i}\omega) \sim C\mathrm{e}^{(1+\mu)\pi\mathrm{i}/4}\omega^{(1+\mu)/2}$. Hence, for the Kelvin-Voigt model and for fractional derivative models, Π is contained in a proper subsector of the right half-plane. For A_3, $\Im\beta(\mathrm{i}\omega) \sim C\,(\Re\beta(\mathrm{i}\omega))^{1/(1-\mu)}$. In particular, for A_3, $\Re\beta(\mathrm{i}\omega) \to \infty$ as $\omega \to \infty$, but Π is not contained in any proper subsector of the right half-plane. Finally, for the standard linear solid model A_4, $\Im\beta(\mathrm{i}\omega) \sim (E+\epsilon\delta)^{-1/2}\omega$ and $\Re\beta(\mathrm{i}\omega) \to \frac{1}{2}\epsilon\delta^2(E+\epsilon\delta)^{-3/2}$.

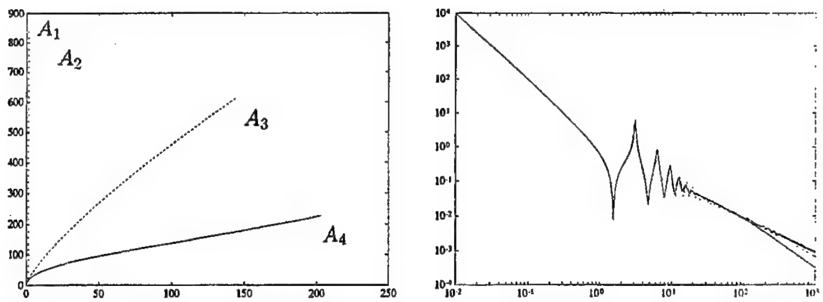

FIG. 1. *Plots of* $\Re\beta(i\omega)$ *(abscissa, left) versus* $Im\beta(i\omega)$ *for* $0 < \omega$ *and of* ω *(abscissa, right) versus* $|\widehat{P}(i\omega)|$.

Plots of the functions $\beta(i\omega)$ and $\widehat{P}(i\omega)$ for the different choices A_1–A_4 for the kernel A are given in Figure 1. (In these plots, and in all later plots, the parameters have been chosen as follows: $E = 1$, $\epsilon = 0.01$, $\delta = 20$, and $\mu = 0.5$.) Note that stronger viscoelastic damping corresponds to a β curve bending more sharply to the right. The same curve style (solid line = A_1, dotted line = A_2, etc.) is used in both graphs, as well as in later graphs of the same type.

4. Rational Approximation of the Plant; First Step. Next we discuss the approximation of the plant in terms of a rational function of β^2, and we begin with the case of the rod with density $\rho(x) \equiv 1$. As we saw in Section 3 formula (17), in this case the transfer function is

$$\widehat{P}(s) = \frac{\beta(s)}{s^2} \coth \beta(s).$$

If, for the moment, we ignore the factor $\beta(s)s^{-2}$, then we are left with the problem of getting an approximation of

$$f(\beta(s)) = \coth \beta(s).$$

As we mentioned earlier, we intend to do this approximation in two steps. First we approximate $f(\beta(s))$ in terms of a rational function of $\beta(s)$, and then we approximate this by a rational function of s.

In the first step, by the argument that we gave earlier, we need a rational approximation of the function f that is valid at least in a sector of the type $\{ z \in \mathbb{C} \mid \arg(z) \leq \pi/4 \}$, and the bigger a sector that we can allow the better. To treat the case where $-A'(0+) < \infty$ we actually need a domain of approximation Π that is asymptotic to a closed half-plane strictly contained in $\Re z > 0$.

There is an obvious candidate for a rational approximation. Recall that

$$f(z) = \coth z = 1/z \prod_{k=1}^{\infty} \frac{1 + (z/\xi_k)^2}{1 + (z/\eta_k)^2},$$

where

$$\xi_k = \pi k - \pi/2, \qquad \eta_k = \pi k,$$

and the convergence is uniform on compact subsets of the complex plane, not containing any of the poles $\pm i\eta_k$ of coth. However, we need convergence in an unbounded domain, and we need fairly explicit estimates on the error, so that we know what happens after we multiply the function by some appropriate weight functions. Observe that if we define a finite approximation f_N by

$$f_N(z) = \frac{1}{z} \prod_{k=1}^{N} \frac{1 + (z/\xi_k)^2}{1 + (z/\eta_k)^2},$$

then the relative error is

$$f(z)/f_N(z) = \prod_{k=N+1}^{\infty} \frac{1 + (z/\xi_k)^2}{1 + (z/\eta_k)^2} = \left(\prod_{k=N+1}^{\infty} \frac{1 + iz/\xi_k}{1 + iz/\eta_k} \right) \times \left(\prod_{k=N+1}^{\infty} \frac{1 - iz/\xi_k}{1 - iz/\eta_k} \right).$$

Actually, as we shall see in a moment, this is not the best possible approximation; to improve the convergence at infinity one needs one minor correction.

Our estimates on the error are based on the following result:

LEMMA 4.1. *For each positive integer N define*

$$g_N(z,c,d) = \prod_{k=N}^{\infty} \frac{1 + z/(\pi(k+c))}{1 + z/(\pi(k+d))}$$

provided that neither $N-1+c$ nor $N-1+d+z/\pi$ is a negative integer. Then

$$\begin{aligned} g_N(z,c,d) &= \frac{\Gamma(N+c)\Gamma(z/\pi+N+d)}{\Gamma(N+d)\Gamma(z/\pi+N+c)} \\ &= \frac{\sin(z+\pi c)\Gamma(N+c)\Gamma(1-z/\pi-N-c)}{\sin(z+\pi d)\Gamma(N+d)\Gamma(1-z/\pi-N-d)}, \end{aligned}$$

where for the second equality we have the additional requirement that neither $c+z/\pi$ nor $d+z/\pi$ is an integer.

Proof. We start with the "well-known" (see, e.g., [7, p. 6]) elementary formula

$$\prod_{k=1}^{\infty} \frac{(k+a_1)(k+a_2)}{(k+b_1)(k+b_2)} = \frac{\Gamma(1+b_1)\Gamma(1+b_2)}{\Gamma(1+a_1)\Gamma(1+a_2)} \tag{27}$$

which is valid provided that

$$a_1 + a_2 = b_1 + b_2$$

and neither b_1 nor b_2 is a negative integer.

To get the first equality, rewrite $g_N(z,c,d)$ in the form

$$\prod_{k=1}^{\infty} \frac{(k+N-1+d)(k+N-1+c+z/\pi)}{(k+N-1+c)(k+N-1+d+z/\pi)}$$

and use (27). For the second equality, use the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad (z \text{ not an integer}). \tag{28}$$

□

We need some estimates on the function $g_N(z,c,d)$ in the Lemma 4.1 of the following type:

LEMMA 4.2. *The following estimates are valid:*

$$\begin{aligned} \frac{\Gamma(z+c)}{\Gamma(z+d)} &= (z+c/2+d/2-1/2)^{c-d} \times \\ &\quad \left(1 + O\left((z+c/2+d/2-1/2)^{-2}\right)\right), \\ &\quad |\arg(z+c)| \le \pi - \epsilon, \quad \epsilon > 0, \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\Gamma(z+c)}{\Gamma(z+d)} &= (-z-c/2-d/2+1/2)^{c-d} \frac{\sin \pi(z+d)}{\sin \pi(z+c)} \times \\ &\quad \left(1 + O\left((-z-c/2-d/2+1/2)^{-2}\right)\right), \\ &\quad |\arg(1-z-d)| \le \pi - \epsilon, \quad \epsilon > 0. \end{aligned} \tag{30}$$

Proof. For a proof for the first part, and for additional terms in the expansion, see [14, p. 34, formula (14)], and to get the second part, use the first part together with (28). □

Now let us apply this to the rod of constant unit density. In this case we get

$$f(z)/f_N(z) = \frac{\Gamma(N+1/2)^2\Gamma(\mathrm{i}z/\pi+N+1)\Gamma(-\mathrm{i}z/\pi+N+1)}{\Gamma(N+1)^2\Gamma(\mathrm{i}z/\pi+N+1/2)\Gamma(-\mathrm{i}z/\pi+N+1/2)}.$$

If we expand the functions above as $z \to \infty$, $z > 0$, by using the asymptotic formulae (29), we observe that for large values of z the ratio $f(z)/f_N(z)$ behaves like $(1+z^2/(\pi(N+1/4)^2)^{1/2}$. Thus, we get a much better approximation if, instead of using the approximation that we originally proposed to use, we divide by this square root. Thus, the approximation that we end up with for $\widehat{C}$ after this first stage is

$$\begin{aligned}\widehat{C}_N^{(1)}(s) &= \frac{\widehat{T}(s)}{\widehat{S}(s)\widehat{P}_N^{(1)}(s)} \\ &= \frac{\widehat{T}(s)}{\widehat{S}(s)}\frac{s^2}{(1+\beta^2/(\pi(N+1/4))^2)^{1/2}}\prod_{k=1}^{N}\frac{1+\beta^2/(\pi k)^2}{1+\beta^2/(\pi(k-1/2))^2}\end{aligned} \tag{31}$$

which leads to a relative error

$$\begin{aligned}\tau_N^{(1)}(s) = \frac{\widehat{C}_N^{(1)}(s)}{\widehat{C}(s)} = &\frac{1}{(1+\beta^2/(\pi(N+1/4))^2)^{1/2}}\times \\ &\frac{\Gamma(N+1/2)^2\Gamma(\mathrm{i}\beta/\pi+N+1)\Gamma(-\mathrm{i}\beta/\pi+N+1)}{\Gamma(N+1)^2\Gamma(\mathrm{i}\beta/\pi+N+1/2)\Gamma(-\mathrm{i}\beta/\pi+N+1/2)}.\end{aligned} \tag{32}$$

for the rod with density $\rho \equiv 1$. The asymptotic estimates in Lemma 4.2 can now be used to estimate the relative error and we obtain

THEOREM 4.3. *Let $A(t)$ be as in Section 3 and let $\tau_N^{(1)}$ be the relative error for the rod with constant density $\rho(x) \equiv 1$ defined in equation (32) above.*

1. *For s in a compact subset K of $\Re s \geq 0$,*

$$\tau_N^{(1)}(s) = 1 + O(N^{-2}) \quad as\ N \to \infty. \tag{33}$$

2. *If β maps $\Re s \geq 0$ into a region that is asymptotically contained in a sector $|\arg z| < \frac{\pi}{2} - \epsilon$, $\epsilon > 0$, (recall that this implies $A(0+) = \infty$), then*

$$\|\tau_N^{(1)} - 1\|_{H^\infty} = O(N^{-2}) \quad as\ N \to \infty. \tag{34}$$

3. *If $-A'(0+) = \infty$, then*

$$\|\tau_N^{(1)} - 1\|_{H^\infty} \to 0 \quad as\ N \to \infty. \tag{35}$$

If, in addition, $A(0+) < \infty$, then we have the more precise estimate

$$\|\tau_N^{(1)} - 1\|_{H^\infty} = O\left((\Re\beta(\mathrm{i}N))^{-2}\right) \quad as\ N \to \infty. \tag{36}$$

4. If $-A'(0) < \infty$, then

$$\limsup_{N\to\infty} \|\tau_N^{(1)}\|_{H^\infty} < \infty \text{ and } \limsup_{N\to\infty} \|1/\tau_N^{(1)}\|_{H^\infty} < \infty. \tag{37}$$

In fact, if $W(s)$ is any weight function with $|W(s)| \to 0$ as $|s| \to \infty$ in $\Re s \geq 0$, then

$$\|W(\tau_N^{(1)} - 1)\|_{H^\infty} \to 0 \text{ and } \|W(1/\tau_N^{(1)} - 1)\|_{H^\infty} \to 0 \text{ as } N \to \infty. \tag{38}$$

In addition,

$$\begin{aligned} \limsup_{|\omega|\to\infty} |\tau_N^{(1)}(i\omega)| &= \frac{1+C}{1-C}\left(1 + O(N^{-2})\right), \\ \liminf_{|\omega|\to\infty} |\tau_N^{(1)}(i\omega)| &= \frac{1-C}{1+C}\left(1 + O(N^{-2})\right) \end{aligned} \tag{39}$$

as $N \to \infty$ where

$$C = \exp\left(A'(0+)/A(0+)^{3/2}\right). \tag{40}$$

Proof. We use Lemma 4.2 to estimate the right hand side of (32) as follows. First, formula (29) with $c = 1/2$ and $d = 1$ yields

$$\frac{\Gamma(N+1/2)}{\Gamma(N+1)} = (N + 1/4)^{-1/2}\left[1 + O(N^{-2})\right] \text{ as } N \to \infty. \tag{41}$$

Since the region Π is symmetric with respect to the real axis, it suffices to consider the case where s is in the first quadrant $Q = \{\, s \mid \Re s \geq 0, \Im s \geq 0 \,\}$.

If $s \in Q$, then $\beta(s) \in Q$ and hence $\Re(-i\beta/\pi + N + 1) \geq N$ for $s \in Q$. Thus by (29) with $c = N + 1$ and $d = N + 1/2$,

$$\frac{\Gamma(-i\beta/\pi + N + 1)}{\Gamma(-i\beta/\pi + N + 1/2)} = (-i\beta/\pi + N + 1/4)^{1/2} \times \left[1 + O(N^{-2})\right] \text{ as } N \to \infty, \tag{42}$$

where the O term holds uniformly for $s \in Q$.

It remains to estimate the difficult factor in (32), namely the ratio

$$\frac{\Gamma(i\beta/\pi + N + 1)}{\Gamma(i\beta/\pi + N + 1/2)}.$$

If s lies in a compact subset K of Q, or if the region Π is asymptotically contained in a sector $|\arg z| \leq \frac{\pi}{2} - \epsilon$ for some $\epsilon > 0$, we can again use (29) with $c = N + 1$ and $d = N + 1/2$ to obtain

$$\frac{\Gamma(i\beta/\pi + N + 1)}{\Gamma(i\beta/\pi + N + 1/2)} = (i\beta/\pi + N + 1/4)^{1/2} \times \left[1 + O(N^{-2})\right] \text{ as } N \to \infty, \tag{43}$$

for all $s \in K$, or for all Q in case the region Π is asymptotically contained in $|\arg z| \leq \frac{\pi}{2} - \epsilon$. In the latter case the constant in the O term depends on how close ϵ is to zero. Combining (41), (42) and (43), completes the proof of Parts 1 and 2 of Theorem 4.3.

When $A'(0+) = -\infty$, but the region Π is not contained in any proper subsector of the right half-plane, we again use (29) with $c = N+1, d = N+1/2$ and $\epsilon = \pi/2$ to see that as $N \to \infty$

$$(44)\quad \frac{\Gamma(\mathrm{i}\beta/\pi + N + 1)}{\Gamma(\mathrm{i}\beta/\pi + N + 1/2)} = (\mathrm{i}\beta/\pi + N + 1/4)^{1/2} \times \left[1 + O\left((\mathrm{i}\beta/\pi + N + 1/4)^{-2}\right)\right]$$

uniformly for those s for which $\Re(\mathrm{i}\beta(s)/\pi + N + 1) \geq 0$. If, on the other hand $\Re(\mathrm{i}\beta/\pi + N + 1) \leq 0$, we use (30) with $c = N+1, d = N+1/2$ and $\epsilon = \pi/2$ to obtain

$$(45)\quad \begin{aligned} \frac{\Gamma(\mathrm{i}\beta/\pi + N + 1)}{\Gamma(\mathrm{i}\beta/\pi + N + 1/2)} &= (-\mathrm{i}\beta/\pi - N - 1/4)^{1/2} \frac{\sin(\mathrm{i}\beta + (N+1/2)\pi)}{\sin(\mathrm{i}\beta + (N+1)\pi)} \\ &\quad \times \left[1 + O\left((-\mathrm{i}\beta/\pi - N - 1/4)^{-2}\right)\right] \\ &= (-\mathrm{i}\beta/\pi - N - 1/4)^{1/2} \mathrm{e}^{\mathrm{i}\pi/2} \left(\frac{1 + \mathrm{e}^{-2\beta}}{1 - \mathrm{e}^{-2\beta}}\right) \\ &\quad \times \left[1 + O\left((-\mathrm{i}\beta/\pi - N - 1/4)^{-2}\right)\right] \\ &= (\mathrm{i}\beta/\pi + N + 1/4)^{1/2} \left(\frac{1 + \mathrm{e}^{-2\beta}}{1 - \mathrm{e}^{-2\beta}}\right) \\ &\quad \times \left[1 + O\left((\mathrm{i}\beta/\pi + N + 1/4)^{-2}\right)\right], \end{aligned}$$

as $N \to \infty$. Here the branch cut has been chosen to make $(\mathrm{i}\beta/\pi + N + 1/4)^{1/2}$ continuous for $\Re s \geq 0$. Combining (44) and (45) with (41) and (42) we find that for $s \in Q$,

$$(46)\qquad \tau_N^{(1)}(s) = 1 + O(N^{-2}) + O\left((\mathrm{i}\beta/\pi + N + 1/4)^{-2}\right) \text{ as } N \to \infty$$

when $\Re(\mathrm{i}\beta/\pi + N + 1) \geq 0$, and

$$(47)\qquad \tau_N^{(1)}(s) = \frac{1 + \mathrm{e}^{-2\beta}}{1 - \mathrm{e}^{-2\beta}} \left(1 + O(N^{-2}) + O\left((\mathrm{i}\beta/\pi + N + 1/4)^{-2}\right)\right) \text{ as } N \to \infty$$

when $\Re(\mathrm{i}\beta/\pi + N + 1) \leq 0$. By Part 3 of Lemma 3.2, $\Re\beta(\mathrm{i}\omega) \to \infty$ as $\Im\beta(\mathrm{i}\omega) \to \infty$ when $-A'(0+) = \infty$, and it follows from (46) and (47) that $\|\tau_N^{(1)} - 1\|_{H^\infty} \to 0$ as $N \to \infty$ as asserted in (35).

For the more precise estimate (36), note first that Part 1 above permits us to restrict attention to a subset $Q' = Q \setminus K$, K compact, and use the asymptotic estimate and inequalities (23) and (24) of Part 2 of Lemma 3.2. By Part 2, we may also assume that $\pi/4 < \arg\beta(s) < \pi/2$ for $s \in Q'$. Thus, for each $s \in Q'$, there exists $\omega(s) \in \mathbf{R}$ with

$$(48)\qquad \Im\beta(\mathrm{i}\omega(s)) = \Im\beta(s), \quad \Re\beta(\mathrm{i}\omega(s)) \leq \Re\beta(s).$$

Let ω_N denote any positive number with $\Im\beta(\mathrm{i}\omega_N) = N\pi$. By (23) and (24) it is sufficient to prove (36) with $\Re\beta(\mathrm{i}N)$ on the right side replaced by $\Re\beta(\mathrm{i}\omega_N)$.

For those s in Q' with $\Im\beta(s) \geq (N+1)\pi$ (i.e., where we use (47)), we get from (48) and (23), (24), the uniform estimate

$$\frac{\Re\beta(\mathrm{i}\omega_N)}{\Re\beta(s)} \leq 1 + o(1) \qquad (N \to \infty). \tag{49}$$

It follows that

$$\left|\frac{1+\mathrm{e}^{-2\beta(s)}}{1-\mathrm{e}^{-2\beta(s)}}\right| = 1 + O\left(\mathrm{e}^{-2\Re\beta(\mathrm{i}\omega_N)}\right) \tag{50}$$

as $N \to \infty$, uniformly for $s \in Q'$ satisfying $\Re(\mathrm{i}\beta(s)/\pi + N + 1) \leq 0$.

Next, note that

$$|\mathrm{i}\beta(s)/\pi + N + 1/4| \geq \max\{\Re\beta(s)/\pi,\ |-\Im\beta(s)/\pi + N + 1/4|\}. \tag{51}$$

Also observe that by Part 2 of Lemma 3.2,

$$\Re\beta(\mathrm{i}\omega_N) = o(N) \qquad (N \to \infty). \tag{52}$$

Thus, unless

$$|-\Im\beta(s)/\pi + N + 1/4| \geq \Re\beta(\mathrm{i}\omega_N), \tag{53}$$

we will have $1/2 \leq \omega(s)/\omega_N \leq 2$ by (23), and hence, by (24),

$$\frac{\Re\beta(\mathrm{i}\omega_N)}{4+o(1)} \leq \Re\beta\left(\mathrm{i}\omega(s)\right) \leq \Re\beta(s). \tag{54}$$

Combining (51), (53) and (54) gives us

$$|\mathrm{i}\beta(s)/\pi + N + 1/4|^{-1} = O\left((\Re\beta(\mathrm{i}\omega_N))^{-1}\right) \quad (N \to \infty) \tag{55}$$

uniformly in Q', and (36) follows from (46), (47), (50), (52) and (55). This completes the proof of Part 3 of Theorem 4.3.

Finally, if $-A'(0+) < \infty$, then Part 3 of Lemma 3.2 gives $\Im\beta(\mathrm{i}\omega) \sim \omega A(0+)^{-1/2}$ and $\Re\beta(\mathrm{i}\omega) \to -\frac{1}{2}A'(0+)/A(0+)^{3/2} > 0$ as $\omega \to \infty$. In particular, the curve $\mathrm{i}\beta(\mathrm{i}\omega)/\pi$ ($\omega > 0$) lies in the second quadrant but remains outside some strip $\{\,\Re z \leq -m,\ 0 \leq \Im z \leq -A'(0+)/4A(0+)^{3/2}\}$; by the argument principle, $\{\,\mathrm{i}\beta(s)/\pi \mid s \in Q\,\}$ is disjoint from this strip as well. It follows that when $N > m+1$,

$$|\mathrm{i}\beta(s)/\pi + N + 1/4| \geq M \qquad (s \in Q)$$

and that

$$\left|\frac{1+e^{-2\beta(s)}}{1-e^{-2\beta(s)}}\right| \leq M$$

for those $s \in Q$ with $\Re(\mathrm{i}\beta(s)/\pi + N + 1) \leq 0$. By (46) and (47), we get the first inequality in (37). For $1/\tau_N^{(1)}$, we apply the asymptotics of Lemma 4.2 to the reciprocal of (32):

this leads to analogues of (46) and (47) from which the bound on $1/\tau_N^{(1)}$ in (37) can be deduced as above. Combining (37) with (33) we see that (38) holds whenever W is a weight with $|W(s)| \to 0$ as $s \to \infty$ in $\Re s \geq 0$. The maximum oscillation estimate (39) is an immediate consequence of (47) and the fact that $\Re\beta(\mathrm{i}\omega) \to -\frac{1}{2}A'(0+)A(0+)^{-3/2}$ as $\omega \to \infty$. This completes the proof of Part 4 of Theorem 4.3. □

For the particular viscoelastic models $A_1 - A_4$ in Section 3, we have

COROLLARY 4.4. *Let $\tau_N^{(1)}$ be the relative error for the rod with constant density $\rho(x) \equiv 1$ defined in equation (32). Then*

1. *For the Kelvin-Voigt model $\widehat{A} = \widehat{A}_1$ and for the fractional derivative model $\widehat{A} = \widehat{A}_2$,*

$$\|\tau_N^{(1)} - 1\|_{H^\infty} = O(N^{-2}) \text{ as } N \to \infty. \tag{56}$$

   Here the constant in the O estimate depends on whether one has A_1 or A_2 and, in the latter case, on the order $1-\mu$ of the fractional derivative.

2. *For the intermediate model $\widehat{A} = \widehat{A}_3$ of order $1-\mu$,*

$$\|\tau_N^{(1)} - 1\|_{H^\infty} = O(N^{-2(1-\mu)}) \text{ as } N \to \infty. \tag{57}$$

3. *For the standard linear solid $\widehat{A} = \widehat{A}_4$,*

$$\begin{aligned} \limsup_{|\omega|\to\infty} |\tau_N^{(1)}(\mathrm{i}\omega)| &= \frac{1+C}{1-C}\left(1 + O(N^{-2})\right), \\ \liminf_{|\omega|\to\infty} |\tau_N^{(1)}(\mathrm{i}\omega)| &= \frac{1-C}{1+C}\left(1 + O(N^{-2})\right) \end{aligned} \tag{58}$$

   as $N \to \infty$ where $C = \exp\left(-\delta^2\epsilon(E+\delta\epsilon)^{-3/2}\right)$.

Proof. Parts (1) and (3) are immediate consequences of Theorem 4.3 and the discussion following Lemma 3.3. The size of the constant in the O term in (56) depends on the sector opening, that is, on the order $1-\mu$ of the fractional derivative. For Part 2, we use the fact that for $\widehat{A} = \widehat{A}_3$, $\Re\beta(\mathrm{i}\omega) \sim C(\Im\beta(\mathrm{i}\omega))^{1-\mu} \sim C(\omega)^{1-\mu}$ as is noted in the discussion following Lemma 3.3. □

REMARK 4.5. *For the general case of constant density $\rho(x) \equiv \rho$, the transfer function for our rod problem is*

$$\widehat{P}(s) = \frac{1}{\rho}\frac{\left(\sqrt{\rho}\beta(s)\right)}{s^2}\coth\left(\sqrt{\rho}\beta(s)\right).$$

In this case the appropriate first-stage relative error $\tau_N^{(1)}$ is defined as in equation (32) above, except that each occurrence of β is replaced by $(\sqrt{\rho}\beta)$. An examination of the proofs (the key difference is that the term $(1+\mathrm{e}^{-2\beta})(1-\mathrm{e}^{-2\beta})^{-1}$ in formula (47) becomes $(1+\mathrm{e}^{-2\sqrt{\rho}\beta})(1-\mathrm{e}^{-2\sqrt{\rho}\beta})^{-1}$), shows that Theorem 4.3 and Corollary 4.4 remain true for the rod with $\rho(x) \equiv \rho$ with the constants C now given by

$$C = \exp\left(\sqrt{\rho}A'(0+)/A(0+)^{3/2}\right) \text{ and } C = \exp\left(-\sqrt{\rho}\delta\epsilon(E+\delta\epsilon)^{-3/2}\right), \tag{59}$$

respectively.

In order to treat the rod with general densities (as well as the beam, see [10]) we need to take account of the fact that the zeros and poles of the transfer function are not spaced at exactly equal intervals. Recall that for the rod with general density $\rho(x)$, the transfer function $\widehat{P}$ is given by $\widehat{P}(s) = \beta(s)s^{-2}f(\beta(s))$ where f, defined in Proposition 3.1, has zeros $\pm i\xi_k$ and nonzero poles $\pm i\eta_k$ satisfying

$$\xi_k = \frac{(2k-1)\pi}{2\rho_1} + \mu_k, \qquad \eta_k = \frac{k\pi}{\rho_1} + \nu_k \qquad (k \to \infty). \tag{60}$$

Here $\rho_1 = \int_0^1 \rho(x)^{1/2}\,dx$, and the perturbation terms μ_k and ν_k satisfy the estimate

$$\mu_k = O(1/k), \qquad \nu_k = O(1/k) \qquad (k \to \infty). \tag{61}$$

We assume that the ξ_k and η_k are known to any degree of accuracy for $1 \le k \le N$, and we define the first-stage approximation to $\widehat{P}$ by

$$\widehat{P}_N^{(1)}(s) = \frac{1}{\rho_0 s^2}\left(1 + (\rho_1\beta)^2/(\pi(N+1/4))^2\right)^{1/2} \prod_{k=1}^{N} \frac{1+(\beta/\xi_k)^2}{1+(\beta/\eta_k)^2},$$

where, by Proposition 3.1, $\rho_0 = \int_0^1 \rho(x)\,dx$. Then the first-stage approximate compensator $\widehat{C}_N^{(1)}$ for the optimal compensator $\widehat{C}(s) = \widehat{T}(s)/(\widehat{S}(s)\widehat{P}(s))$ is given by $\widehat{C}_N^{(1)}(s) = \widehat{T}(s)/(\widehat{S}(s)\widehat{P}_N^{(1)}(s))$, and the relative error is

$$\tau_N^{(1)}(s) = \frac{1}{(1+(\rho_1\beta)^2/(\pi(N+1/4))^2)^{1/2}} \prod_{k=N+1}^{\infty} \frac{1+(\beta/\xi_k)^2}{1+(\beta/\eta_k)^2}. \tag{62}$$

In order to estimate the relative error $\tau_N^{(1)}(s)$, we set

$$\xi_k' = \frac{(2k-1)\pi}{2\rho_1} \text{ and } \eta_k' = \frac{k\pi}{\rho_1}, \qquad (k \to \infty), \tag{63}$$

and write

$$\begin{aligned} \tau_N^{(1)}(s) =& \frac{1}{(1+(\rho_1\beta)^2/(\pi(N+1/4))^2)^{1/2}} \prod_{k=N+1}^{\infty} \frac{1+(\beta/\xi_k')^2}{1+(\beta/\eta_k')^2} \\ &\times \prod_{k=N+1}^{\infty} \frac{1+(\beta/\xi_k)^2}{1+(\beta/\xi_k')^2} \times \prod_{k=N+1}^{\infty} \frac{1+(\beta/\eta_k')^2}{1+(\beta/\eta_k)^2} \\ \equiv& \tilde{\tau}_N^{(1)}(s) \times \prod_{k=N+1}^{\infty} \frac{1+(\beta/\xi_k)^2}{1+(\beta/\xi_k')^2} \times \prod_{k=N+1}^{\infty} \frac{1+(\beta/\eta_k')^2}{1+(\beta/\eta_k)^2}. \end{aligned} \tag{64}$$

Here $\tilde{\tau}_N^{(1)}$ is given by the expression on the right hand side of (32), but with each occurrence of β replaced by $(\rho_1\beta)$. Theorem 4.3 and Remark 4.5 apply to $\tilde{\tau}_N^{(1)}$. The following Lemma yields estimates for the remaining two products in equation (64).

LEMMA 4.6. *Let ψ_k, ψ'_k $(k = 1, 2, \ldots)$ satisfy*

$$\psi_k,\ \psi'_k > 0, \quad |\psi'_k - ck| < b < \infty \tag{65}$$

for fixed positive constants b and c. Write $\delta_k = |\psi'_k - \psi_k|$ and assume that

$$\delta_k = O(1/k) \qquad (k \to \infty). \tag{66}$$

Define $Q_N(z)$ by

$$Q_N(z) = \prod_{k=N+1}^{\infty} \frac{1 + z/\psi_k}{1 + z/\psi'_k}. \tag{67}$$

Then as $N \to \infty$ the following uniform estimates hold:

1. $|Q_N(z) - 1| = O(\sum_{k>N} \delta_k/k^2)$ *as $N \to \infty$ uniformly in compact subsets K of* $\mathbf{C}$.
2. $|Q_N(z) - 1| = O(\sum_{k>N} \delta_k/k)$ *as $N \to \infty$ uniformly in $D_1 = \{ z \mid |\arg z| < \theta < \pi \}$.*
3. $|Q_N(z) - 1| = O(\sum_{k>N} \delta_k/k + \sup_{k>N} \delta_k \log k)$ *as $N \to \infty$ uniformly in $D_2 = \{ z \mid \Re z \geq 0 \} \cup \{ z = x + iy \mid x < 0, |y| > d > 0 \}$.*
4. $|Q_N(z) - 1| = O(\sup_{k>N} \delta_k k^p)$ *as $N \to \infty$ uniformly in $D_3 = \{ z \mid \Re z \geq 0 \} \cup \{ z = x + iy \mid x < 0, |y| > d/(|x|^p + 1) \}$ with $0 < p \leq 1$.*

We remark that for each part of Lemma 4.6, the conclusion is valid provided only that the $\{\delta_k\}$ are such that the O term in the conclusion is finite; this is clearly the case when (66) holds. Part 4 of Lemma 4.6 is not used for our analysis of viscoelastic rods since the region Π is asymptotic to a closed half-plane strictly contained in $\Re z > 0$. We include it here since its proof is a minor modification of that of Part 3, and we will use Part 4 to study the analogous problem for a viscoelastic Euler-Bernoulli beam [10] with $\delta_k = O(e^{-k})$ as $k \to \infty$. This is needed due to the fact that for the case of a standard linear solid Euler-Bernoulli beam, the image of $\Re s \geq 0$ under the appropriate analogue of β approaches the axis at infinity like a hyperbola.

Proof. Throughout the proof M denotes a positive constant whose value may change from one line to the next.

Write

$$\log Q_N(z) = \sum_{k=N+1}^{\infty} \log(1 + t_k(z)),$$

where

$$t_k(z) = \frac{1 + z/\psi_k}{1 + z/\psi'_k} - 1 = \frac{\psi'_k - \psi_k}{\psi_k} \frac{z}{z + \psi'_k}.$$

Since

$$|Q_N - 1| \leq \left| \int_0^{\log Q_N} e^s \, ds \right| \leq |\log Q_N| \exp |\log Q_N|$$

(the integral is the path integral along the straight line joining 0 to $\log Q_N$), it is clearly sufficient to show that the conclusions of Parts 2, 3 and 3 hold with their left sides replaced by

$$T_N(z) \equiv \sum_{k=N+1}^{\infty} |t_k(z)|.$$

But in D_1, $|z(z+\psi'_k)^{-1}|$ is uniformly bounded, so

$$T_N(z) \leq M \sum_{k=N+1}^{\infty} \frac{\delta_k}{k} \qquad (z \in D_1), \tag{68}$$

since $\psi_k \sim ck$, and Part 2 is proved. For z in a compact subset K of C, $|z(z+\psi'_k)^{-1}| = O(k^{-1})$ uniformly as $k \to \infty$, and we obtain an extra factor of k in the denominator of the sum in (68). This proves Part 1.

To prove Part 3, we can restrict attention to $D'_2 \equiv D_2 \backslash D_1(\theta = 3\pi/4)$. For $z = x + iy$ in D'_2, if $\psi'_k \leq -x/2$ or $\psi'_k \geq -3x/2$, then $|z(z+\psi'_k)^{-1}| \leq M$, and we can proceed as above. Let

$$K_N(z) = \{\, k \mid k > N \text{ and } -x/2 < \psi'_k < -3x/2 \,\}, \qquad \Delta_N(z) = \sup_{k \in K_N(z)} \delta_k$$

(with $\Delta_N(z) = 0$ if $K_N(z)$ is empty). Then

$$\begin{aligned} T'_N(z) \equiv \sum_{k \in K_N(z)} |t_k(z)| &\leq M \sum_{k \in K_N(z)} \frac{\delta_k}{|z+\psi'_k|} \\ &\leq M \Delta_N(z) \sum_{k \in K_N(z)} \frac{1}{|z+\psi'_k|} \end{aligned} \tag{69}$$

since $|z/\psi_k|$ is uniformly bounded above and below on $K_N(z)$.

To estimate the sum in (69), we use

$$\begin{aligned} |z+\psi'_k| &\geq \max\{\, |y|,\ |x+\psi'_k| \,\} \\ &\geq \max\{\, |y|,\ |x+ck| - b \,\}. \end{aligned}$$

For fixed z, if $\Delta_N(z) > 0$, there is a $j \in K_N(z)$ such that $\delta_j > \frac{1}{2}\Delta_N(z)$. In particular, $j > N$ and

$$cj + b > -\frac{x}{2} \geq \frac{1}{2\sqrt{2}}|z|.$$

It follows that

$$\begin{aligned} &\Delta_N(z) \sum_{k \in K_N(z)} \frac{1}{|z+\psi'_k|} \\ &\leq 2\delta_j \left(\sum_{\substack{k \in K_N(z) \\ |x+ck| \leq b+c}} \frac{1}{|y|} + \sum_{\substack{k \in K_N(z) \\ |x+ck| > b+c}} \frac{1}{(|x+ck| - b)} \right) \end{aligned}$$

The index set for the first sum contains no more than $2b/c+3$ terms, while the second sum is comparable to the sum of two finite harmonic series. We get

$$\begin{aligned}T_N'(z) \leq & M\delta_j \left(\frac{2b/c+3}{d} + \log|z|\right) \\ \leq & M\delta_j(1+\log j) \leq M \sup_{k>N}\{\delta_k \log k\};\end{aligned} \tag{70}$$

this proves Part 3. The proof of Part 4 is identical, except that (70) becomes

$$\begin{aligned}T_N'(z) & \leq & M\delta_j(|z|^p + 1 + \log|z|) \\ & \leq & M \sup_{k>N}\{\delta_k k^p\}.\end{aligned}$$

This completes the proof of Lemma 4.6. □

Clearly, Part 4 can be generalized from D_3 to other regions where $|y| > d/(1+|x|^p)$ is replaced by $|y| > \mu(|x|)$ with $\mu(|x|) \to 0$ as $|x| \to \infty$.

Combining Lemma 4.6 with Theorem 4.3 and Remark 4.5 we obtain

THEOREM 4.7. *Let $A(t)$ and $\rho(x)$ be as in Section 3 and let $\tau_N^{(1)}$ be the first-stage relative error defined in equation (62) for the rod with density $\rho(x)$. Then Parts 1–4 to Theorem 4.3 remain valid provided that the following modifications are made.*

1. *In Part 2 the estimate (34) must be replaced by the weaker estimate*

$$\|\tau_N^{(1)} - 1\|_{H^\infty} = O(N^{-1}) \text{ as } N \to \infty. \tag{71}$$

2. *In Part 3 the precise estimate (36) that holds when $A(0+) < \infty$ must be replaced by*

$$\|\tau_N^{(1)} - 1\|_{H^\infty} = O\left((\Re\beta(\mathrm{i}N))^{-2}\right) + O\left(N^{-1}\log N\right) \text{ as } N \to \infty. \tag{72}$$

3. *In Part 4 equation (39) must be replaced by*

$$\begin{aligned}\limsup_{|\omega|\to\infty}|\tau_N^{(1)}(\mathrm{i}\omega)| =& \frac{1+C}{1-C}\left(1+O(N^{-1}\log N)\right), \\ \liminf_{|\omega|\to\infty}|\tau_N^{(1)}(\mathrm{i}\omega)| =& \frac{1-C}{1+C}\left(1+O(N^{-1}\log N)\right)\end{aligned} \tag{73}$$

where the constant C is now given by

$$C = \exp\left(\rho_1 A'(0+)/A(0+)^{3/2}\right), \tag{74}$$

with $\rho_1 = \int_0^1 \rho(x)^{1/2}\,\mathrm{d}x$.

Proof. As noted earlier, $\tilde{\tau}_N^{(1)}$ satisfies the conclusions of Theorem 4.3 with formula (40) replaced by (74). The remaining two products in (64) may be rewritten as four products with linear Möbius factors each of which may be estimated by Lemma 4.6.

Specifically, one of these products is

$$Z_N(s) = \prod_{k=N+1}^{\infty} \frac{1+\mathrm{i}\beta/\xi_k}{1+\mathrm{i}\beta/\xi_k'}.$$

Since (61) holds and, by Lemma 3.2, $\Pi_1 \equiv \{\, i\beta(s) \mid \Re s \geq 0 \,\}$ is always asymptotically contained in a region of the form D_2, it follows from Part 3 of Lemma 4.6 that

$$\begin{aligned}
\|Z_N - 1\|_{H^\infty} =& O\left(\sum_{k>N} \frac{\mu_k}{k} + \sup_{k>N} \mu_k \log k\right) \\
=& O\left(\sum_{k>N} k^{-2} + \sup_{k>N} k \log k\right) = O\left(N^{-1}\log N\right) \text{ as } (N \to \infty).
\end{aligned}$$

If Π is asymptotically contained in a sector $\{\, z \mid |z| < \theta < \pi/2 \,\}$, then Part 2 of Lemma 4.6 yields $\|Z_N - 1\|_{H^\infty} = O(N^{-1})$, while Part 1 shows that $Z_N(s) = 1 + O(N^{-2})$ as $N \to \infty$ uniformly for s in a compact subsets K of $\Re s \geq 0$. The other three products of linear Möbius factors which occur in (64) may be treated in exactly the same manner, and Theorem 4.7 follows. $\square$

Of course, the general density version of Corollary 4.4 for the particular viscoelastic models $A_1 - A_4$ also holds after making the same modifications as in Theorem 4.7.

For each of the models, Figure 2 shows $|\tau_N^{(1)}(i\omega) - 1|$ (ordinate), as defined in (32), plotted against ω for N increasing from $N = 5$ to $N = 55$ in increments of 10. The curves move down and to the right as N increases.

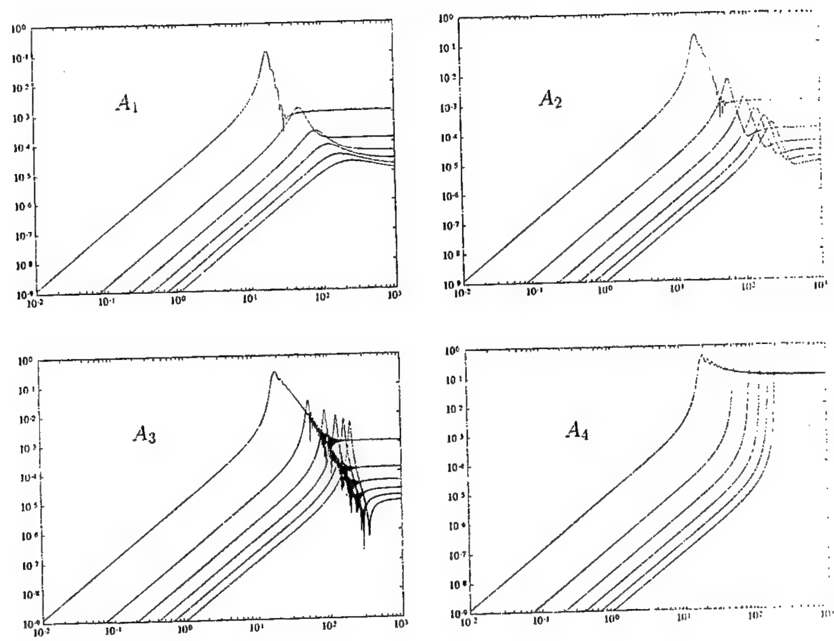

FIG. 2. *Plots of* $|r_N^{(1)}(\mathrm{i}\omega) - 1|$

5. Approximation of Möbius Transforms in β^2 by Möbius Transforms in s. After the first approximation step, we arrive at a finite product of factors (Möbius transformations in β^2) of the type

$$M_k(s) = \frac{1+(\beta/\xi_k)^2}{1+(\beta/\eta_k)^2}.$$

Substituting the definition of β into these factors we get

$$M_k(s) = \frac{s/\xi_k^2 + \widehat{A}(s)}{s/\eta_k^2 + \widehat{A}(s)}.$$

How do we approximate these factors by rational functions of s?

Let us take a closer look at one of these factors, where for simplicity we denote ξ_k by ξ and η_k by η, i.e., we write it in the form

$$M(s) = \frac{s/\xi^2 + \widehat{A}(s)}{s/\eta^2 + \widehat{A}(s)}. \tag{75}$$

As $s \to \infty$ this factor tends to η^2/ξ^2, which is close to 1 (since η_k^2/ξ_k^2 tends to 1 as $k \to \infty$), and as $s \to 0$ it tends to 1 (since $\widehat{A}(s) \sim E/s$ as $s \to 0$). The largest deviation from 1 on the imaginary axis occurs in the cross-over region where the order of magnitude of $\widehat{A}$ is the same as the order of magnitude of s/ξ^2 and s/η^2. This indicates that a rational approximation of $M(s)$ should have its zeros and poles somewhere in the left half-plane close to the cross-over region. As we shall see below, this is true, at least if we restrict our attention to the case where the internal damping is weak enough.

In Section 4 the amount of internal damping, described by the parameter ϵ in our examples A_1–A_4, was significant only in that it affects the size of the O constants in our estimates, and hence the number of terms N needed to get a good first-stage approximation. It did not determine the shape of the image of the half-plane $\Re s \geq 0$ under the map β, and consequently did not affect the order of our convergence estimates. In this Section we shall make use of the fact that in many materials of interest, the damping parameter ϵ is quite small compared to the elastic parameter E.

More precisely, we assume that $\widehat{A}$ is of the form

$$\widehat{A}(s) = E/s + \epsilon\widehat{a}(s), \tag{76}$$

where a has been scaled so that, for example,

$$\widehat{a}(0) = 1.$$

Then (75) can be written in the form

$$M(s) = \frac{s/\xi^2 + E/s + \epsilon\widehat{a}(s)}{s/\eta^2 + E/s + \epsilon\widehat{a}(s)}. \tag{77}$$

For $\epsilon = 0$ (no damping) this function has two purely imaginary zeros located at $\pm\mathrm{i}\sqrt{E}\xi$, and two purely imaginary poles located at $\pm\mathrm{i}\sqrt{E}\eta$. By continuity, for small nonzero values of ϵ, M will have zeros and poles close to these.

As the following lemma shows, these are the only complex zeros and poles of the function M:

LEMMA 5.1. *For each constant $\zeta > 0$, the function $s/\zeta^2 + \widehat{A}(s)$ has at most one pair of complex conjugate zeros.*

For a proof, see Desch and Grimmer [6].

For each ζ and ϵ such that $s/\zeta^2 + E/s + \epsilon\widehat{a}(s)$ has a pair of complex conjugate roots, denote the root of this function in the upper half-plane by $s_{\zeta,\epsilon}$. Then the appropriate first order approximation of $M(s)$ is

$$N(s) = \frac{(1 - s/s_{\xi,\epsilon})(1 - s/\bar{s}_{\xi,\epsilon})}{(1 - s/s_{\eta,\epsilon})(1 - s/\bar{s}_{\eta,\epsilon})}.$$

(We have normalized N so that $N(0) = M(0) = 1$.) Let $N_k(s)$ be the same function with ξ replaced by ξ_k and η replaced by η_k. Then the total relative error that we introduce at this stage is

$$\tau_N^{(2)}(s) \equiv \frac{\widehat{C}_N^{(2)}(s)}{\widehat{C}_N^{(1)}(s)} \equiv \prod_{k=1}^{N} M_k(s)/N_k(s).$$

The analytic estimates that we are able to prove for the size of $\tau_N^{(2)}$ are still incomplete. We have been able to show that, under quite general assumptions, the H^∞-norm of each factor $M_k(s)/N_k(s) - 1$ is of order $O(\epsilon)$ as $\epsilon \to 0$. However, the $O(\epsilon)$-constant that we are able to obtain deteriorates as one multiplies N successive factors and lets $N \to \infty$. This leaves open the question of whether it is in fact true that $\|\tau_N^{(2)} - 1\|_{H^\infty} = O(\epsilon)$ as $\epsilon \to 0$, uniformly in N. Observe that for the Kelvin-Voigt model A_1 each factor $M_k(s)$ can be expressed as the ratio of quadratics in s, so the result is exact; i.e., there is no error introduced at this stage. We are also able to show that if $a(t)$ is a finite sum of exponentially decaying terms (the natural generalization of the standard linear solid model A_4) so that $\widehat{A}(s)$ is a rational function (in which case the approximation of the factors $M_k(s)$ by $N_k(s)$ serves only to lower the order of the approximate compensator), then it is indeed true that $\|\tau_N^{(2)} - 1\|_{H^\infty} = O(\epsilon)$ as $\epsilon \to 0$ uniformly in N. For the models A_2 and A_3 we are forced to simply compute the error numerically.

In Figure 3, $|\tau_N^{(2)}(i\omega) - 1|$ (ordinate) is plotted against ω for each of the models A_2, A_3 and A_4. In each case, N increases from 5 to 55 in increments of 10 from the bottom graph to the top one. Similar plots, not shown here, were made for a more refined procedure where a real third root of the numerator and of the denominator of each $M_k(s)$ (these approach the negative real zero of $\widehat{A}(s)$ closest to the origin as $k \to \infty$) was included in the approximation of $M_k(s)$ to account partially for the creep response. For the parameters used here improvement by up to a factor of 2 in the relative error $\tau_N^{(2)}$ was observed in the irrational cases (and there is then no error at this stage for model A_4).

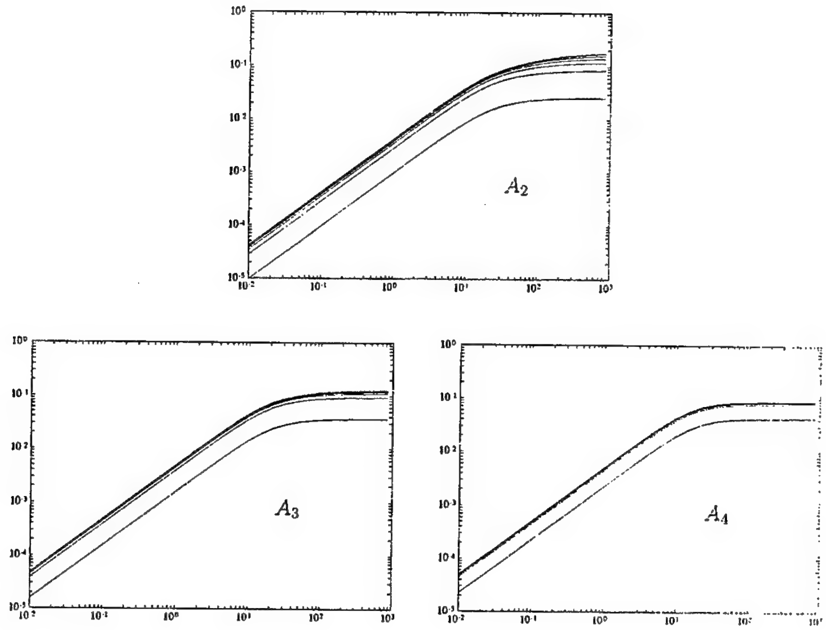

FIG. 3. *Plots of* $|\tau_N^{(2)}(\mathrm{i}\omega) - 1|$

6. The Final Approximation. When we combine the different expansions described in the previous sections we get a rational approximation, except for the fact that there is a left-over irrational factor in the denominator of (31) that has not been accounted for, namely the factor

$$\left(1+\beta^2/(\pi(N+1/4))^2\right)^{1/2}. \tag{78}$$

Here we look at low order approximations of (78). In particular, we investigate the size of the error that one gets by ignoring this factor completely; note that it tends to 1 uniformly on compact subsets of the right half-plane as $N \to \infty$.

The relative errors $\tau_N^{(1)}$ $(N = 1, 2, \ldots)$, together with their reciprocals, are uniformly bounded in H^∞ (Theorems 4.3 and 4.7), and the same is true of the errors $\tau_N^{(2)}$ of Section 5, at least in the special case where $\widehat{A}(s) = E/s + \epsilon\widehat{a}(s)$ is rational and ϵ is small relative to E. This type of boundedness will no longer hold for the new error introduced when the square root in (78) is approximated by a rational function, because in general this root will not have a rational growth rate at infinity. Thus, in the final approximation, in most cases the magnitude of τ_N must be either unbounded, or not bounded away from zero. However, as we saw in Section 2, the only thing that we really have to worry about is that we satisfy the requirement that $\limsup_{N\to\infty}\|\tau_N\widehat{C}\widehat{S}\|_{H^\infty} < \infty$. More precisely, since $\tau_N^{(j)}$ has been uniformly bounded in the preceding sections, if we denote the approximation that we will use for the root in (78) by h_N, then we need only have h_N converging to one uniformly on compact subsets of the right half-plane, and, in addition,

$$\limsup_{N\to\infty}\left\|\tau_N^{(3)}\widehat{C}\widehat{S}\right\|_{H^\infty} < \infty, \tag{79}$$

where

$$\tau_N^{(3)}(s) \equiv \frac{\widehat{C}_N(s)}{\widehat{C}_N^{(2)}(s)} \equiv \frac{(1+\beta(s)^2/(\pi(N+1/4))^2)^{1/2}}{h_N(s)}.$$

At this stage the size of the function $\widehat{C}$ at infinity becomes important. There are two possibilities: either $\widehat{C}(\infty) = 0$ or $\limsup_{N\to\infty}|\widehat{C}(s)| > 0$. (In the former case the ideal compensator is strictly proper, in the latter it is not.)

The second of these possibilities can occur in only one way. Recall that $\widehat{C}(s) \sim \widehat{T}(s)/\widehat{P}(s)\widehat{S}(s) \sim \widehat{T}(s)/\widehat{P}(s)$ as $s \to \infty$. The function $\widehat{T}$ must have a zero of integer order at infinity, since we require it to be rational. Concerning $\widehat{P}(s)$, we have the following estimates from the results of Section 3.

LEMMA 6.1. *Let $A(t)$ be as in Section 3.*

1. *$\widehat{P}(s)$ is analytic in the closed right half-plane, except for a double pole at 0, and*

$$\frac{|s^2\widehat{P}(s)|}{1+|\beta(s)|}$$

is bounded and bounded away from 0 in $\Re s \geq 0$. Moreover,

2. *if $A(0+) < \infty$, then $|s\widehat{P}(s)|$ is bounded and bounded away from 0 as $s \to \infty$ in $\Re s \geq 0$, and*
3. *if $A(0+) = \infty$, then $|\widehat{P}(s)| = o(|s|^{-1})$ and $|\widehat{P}^{-1}(s)| = O\left(|s|^{3/2}\right)$ as $s \to \infty$ in $\Re s \geq 0$.*

Proof. By Lemma 4.6, we need only look at (17), the case of uniform density. Then Part 1 follows from Lemma 3.2, Parts (1b) and 3 (with the argument principle). Parts 2 and 3 are clear from Lemma 3.2, Parts (1g)and (1h). □

Thus, since $\widehat{C}(s)$ must be bounded for large s (see (4)), the only case where $\widehat{C}(\infty) = 0$ fails to hold is the one where $A(0+) < \infty$ and $\widehat{T}$ has a first order zero at infinity. In all other cases $\widehat{T}$ must have at least a second order zero at infinity, and $\widehat{C}(\infty) = 0$.

Let us first discuss the case that in view of the discussion above is more common, namely the one where $\widehat{C}(\infty) = 0$, and $\widehat{T}$ has at least a second order zero at infinity. We claim that in this case we may take $h_N \equiv 1$, i.e., we may ignore the square root completely. This follows from the following lemma:

LEMMA 6.2. *If $\widehat{T}(s) = O(s^{-2})$ as $s \to \infty$, then (79) holds with $h_N \equiv 1$.*

Proof. By Part 1 of Lemma 6.1, it suffices to show that

$$\limsup_{N\to\infty} \sup_{\Re s \geq 0} \frac{|1 + \beta^2/(\pi(N + 1/4))^2|^{1/2}}{1 + |\beta(s)|} < \infty.$$

But this is trivially true since the numerator can be estimated from above by $1 + |\beta(s)|/(\pi(N + 1/4)) < 1 + |\beta(s)|$. □

Note that if we use $h_N \equiv 1$, then the approximate compensators $\widehat{C}_N = \tau_N^{(1)}\tau_N^{(2)}\tau_N^{(3)}\widehat{C}$ will not be strictly proper when $\widehat{T}$ has a second order zero at infinity. If one wants a strictly proper compensator, then one may instead use $h_N(s) = (1+\epsilon_N s)$, where $\epsilon_N \to 0$ as $N \to \infty$; this sequence can be chosen more or less arbitrarily. This does not disturb (79).

Now let us return to the case where $\widehat{C}(\infty) \neq 0$. Clearly, in this case we cannot take $h_N \equiv 1$, since the square root in (78) grows like s as $s \to \infty$, invalidating (79). However, the following modification is sufficient.

LEMMA 6.3. *If $\widehat{T}(s)$ has a first order zero at infinity (and hence $\widehat{C}(\infty) \neq 0$), then (79) holds with $h_N = 1 + \epsilon_N s$, where, e.g., $\epsilon_N = 1/(\pi N + 1/4)$, or more generally, ϵ_N represents any sequence satisfying $\epsilon_N \to 0$ and $\epsilon_N^{-1} = O(N)$ as $N \to \infty$.*

Proof. In this case $\widehat{C}(\infty)\widehat{S}(\infty) \neq 0$, and we must show that

$$\limsup_{N\to\infty} \sup_{\Re s \geq 0} \frac{|1 + \beta^2/(\pi(N + 1/4))^2|^{1/2}}{|1 + \epsilon_N s|} < \infty.$$

This is true, because $|1 + \epsilon_N s| \geq (1 + \epsilon_N|s|)/\sqrt{2}$, and, for some constant $K \geq 1$, $|\beta(s)| \leq K|s|$ and $\epsilon_N \geq 1/(KN)$, and hence

$$\frac{|1 + \beta^2/(\pi(N + 1/4))^2|^{1/2}}{|1 + \epsilon_N s|} \leq \frac{1 + K|s|/(\pi N + 1/4)}{\sqrt{2}(1 + |s|/(KN))},$$

the supremum of which over s stays bounded as $N \to \infty$. □

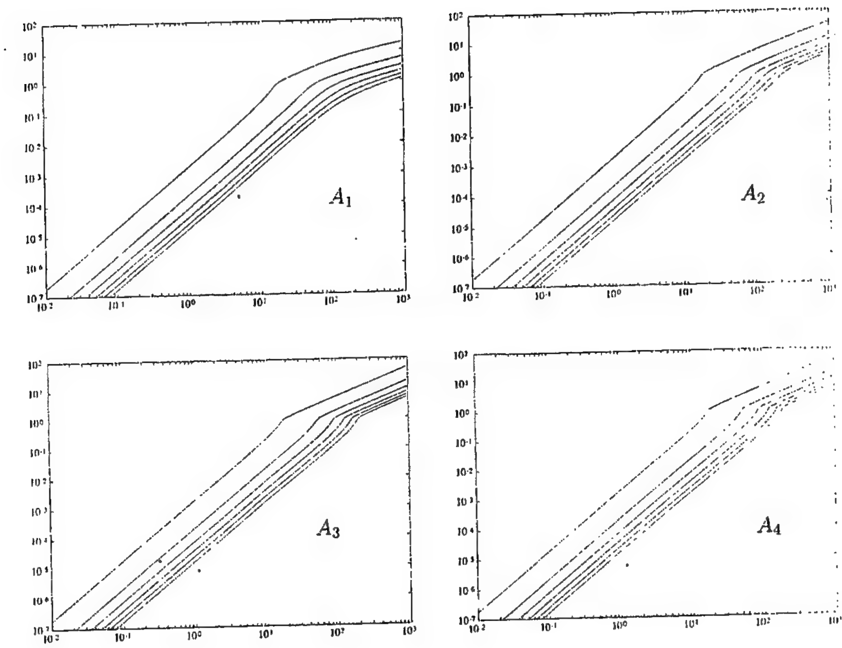

FIG. 4. *Plots of* $|\tau_N^{(3)}(i\omega) - 1|$

Note that, in contrast to the error $\tau_N^{(2)}$ introduced in Section 5, $\tau_N^{(3)}(s)$ approaches one as $N \to \infty$; there is no residual error. Nonetheless, at least in our examples, the error $\tau_N^{(3)}$ will be the dominating error for high frequencies.

We plot $|\tau_N^{(3)}(i\omega) - 1|$ (for N between 5 and 55, increasing to the right) versus ω in Figure 4.

To produce some pictures of the final result we have fixed the "ideal" sensitivity function $\widehat{S}$ by choosing

$$\widehat{S}(s) = \left(1 + \frac{b(s+\epsilon)}{s^2(1+\epsilon s)}\right)^{-1},$$

where $b = 5$ and $\epsilon = 0.01$. Then $\widehat{T} = 1 - \widehat{S}$ has a second order zero at infinity. Here b determines the cross-over frequency, and ϵ restricts the size of $\widehat{P}\widehat{S}$ at zero and the size of $\widehat{C}\widehat{S}$ at infinity; cf. (4). A plot of $|\widehat{S}|$ and $|\widehat{T}|$ is given in Figure 5, as well as plots of $|\widehat{C}|$ for different choices of kernels. Plots of $|\widehat{P}\widehat{S}|$ and $|\widehat{C}\widehat{S}|$, which pertain to the stability requirement (4), are given in Figure 6. The sizes of $|\widehat{S}\widehat{C}\tau_N|$ (see (13)) are plotted in Figure 7, and the error $|\tau_N - 1|$ weighted by $|\widehat{T}|$ is plotted in Figure 8. Here τ_N is the

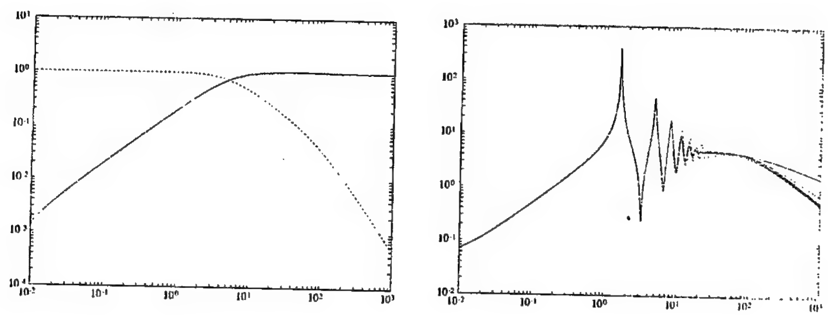

FIG. 5. *Plots of* $|\widehat{S}|$, $|\widehat{T}|$ *(left,* $\widehat{S}(\infty) = 1$*) and* $|\widehat{C}|$

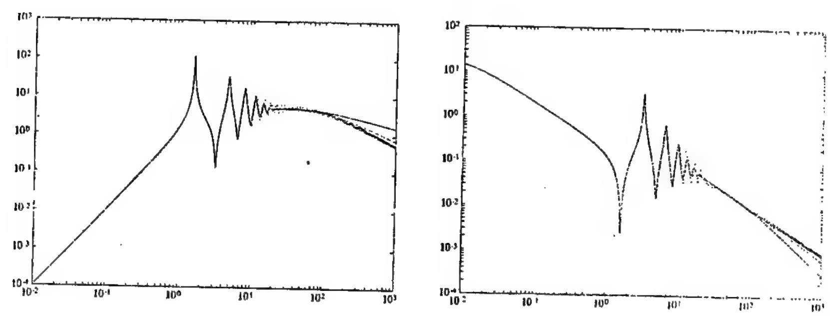

FIG. 6. *Plots of* $|\widehat{C}\widehat{S}|$ *(left) and* $|\widehat{P}\widehat{S}|$

total relative error $\tau_N = \tau_N^{(1)}\tau_N^{(2)}\tau_N^{(3)}$. (As above, the ω axis is horizontal.)

These plots summarize the combined effect of all the approximations. Concerning Figure 7, note that the weighting by $\widehat{C}\widehat{S}$ controls the unboundedness of τ_N at high frequencies (due to $\tau_N^{(3)}$) to the extent that the validity of (13) is determined at the first peak of the graph ($\omega \sim 1.5$). In Figure 8, observe that for A_2, A_3 and A_4, the curves rise to a limit in the lower frequencies as N increases; this residual error reflects the neglected creep response in the approximation of Section 5. For A_1, the Möbius factors of Section 5 can be expressed as quotients of second degree polynomials in s, so no approximation is necessary and we see no "folding over" in the figure. For A_4, we get cubic polynomials; by omitting the approximation step of Section 5 we end up with a graph (not shown) similar to the one for A_1 above at low frequencies.

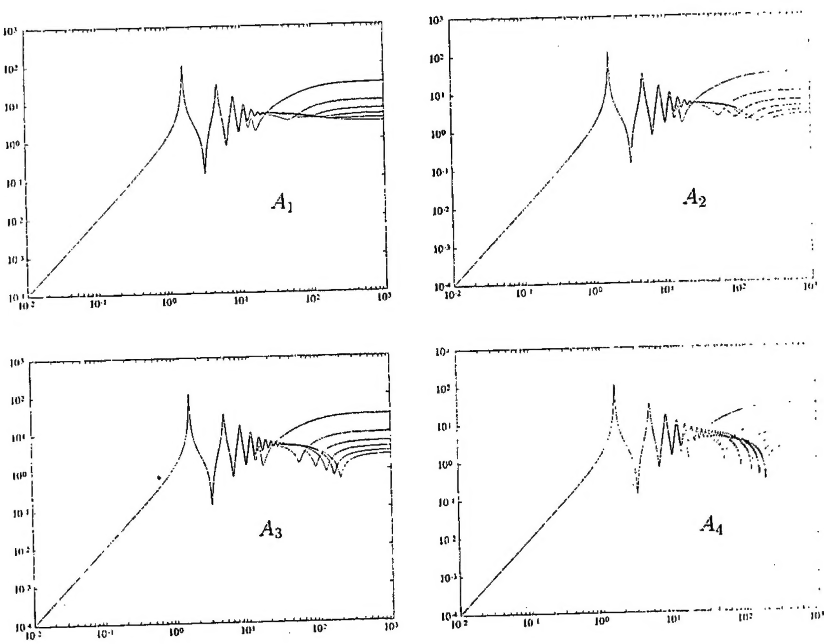

FIG. 7. *Plots of* $|\tau_N \widehat{C}\widehat{S}|$ *for the final (rational) approximation* ($5 \le N \le 55$)

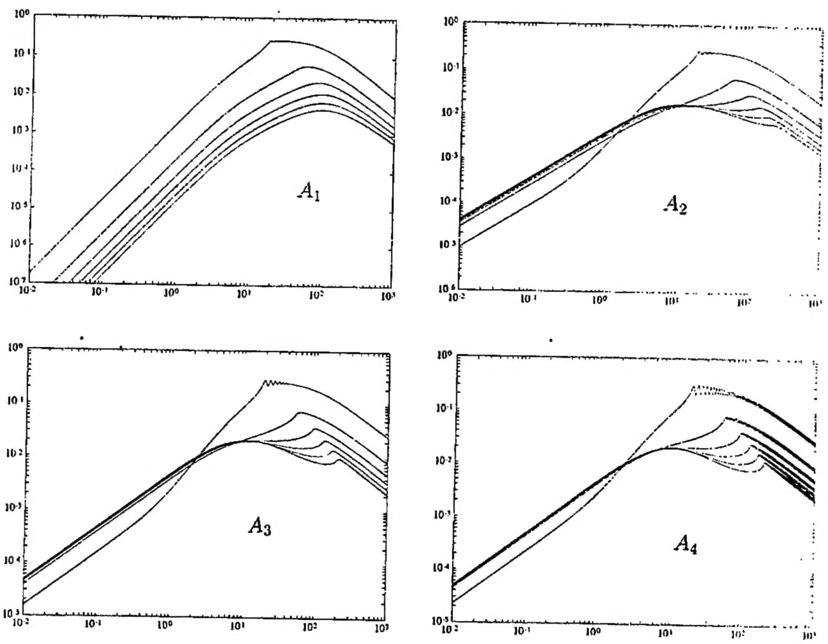

FIG. 8. *Plots of* $|(\tau_N(i\omega) - 1)\widehat{T}(i\omega)|$ *for the final approximation*

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